UNIVERSITY OF TORONTO

Faculty of Arts and Science

term test #2, Version 2

CSC236F

Date: Friday November 16, 11:10–12:00pm or 12:10–1:00pm

Duration: 50 minutes

Instructor(s): Danny Heap

Examination Aids: pencils, pens, erasers, drinks, snacks

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first and last names:

utorid:

student number:

Please read the following guidelines carefully!

- **Please write your name, utorid, and student number on the front of this exam.**
- This examination has **2** questions. There are a total of **5 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely.
- You will receive 20% of the marks for any question you leave blank or indicate "I cannot answer this question."

Take a deep breath.
This is your chance to show us
How much you've learned.

We **WANT** to give you the credit

Good luck!

1. **[13 marks]** ($\approx$ 35 minutes)

Define $T(n)$ by:

$$T(n) = \begin{cases} 1 & \text{if } n < 3 \\ 1 + 2T(n-3) & \text{if } n \geq 3 \end{cases}$$

(a) **[3 marks]** Let $q \in \mathbb{N}$, and let $r \in \{0, 1, 2\}$. Use the method of repeated substitution (unwinding) to find a conjecture for a closed form for $T(3q + r)$, that is some function c, using a fixed number of elementary operations, such that $c(3q + r) = T(3q + r)$. You may assume that if $r \in \mathbb{R}, n \in \mathbb{N}$, and $r \neq 1$, then $\sum_{i=0}^{i=n} r^i = (r^{n+1} - 1)/(r - 1)$, if that assumption turns out to be useful.

sample solution: I notice that if $3q + r - 3 = 3(q-1) + r$, and that $T(r) = 1$, so assuming $q \geq 1$ for the moment:

$$\begin{aligned} T(3q+r) &= 1 + 2T(3(q-1)+r) = 1 + 2(1 + 2T(3(q-2)+r)) \qquad \text{\# by definition of } T \\ &= 2^0 + 2^1 + 2^2 T(3(q-2)+r) \\ &= \\ &= \vdots \text{ (I see a pattern)} \\ &= 2^0 + 2^1 + 2^2 + \cdots + 2^q T(3(q-q)+r) = \sum_{i=0}^{i=1} 2^i = 2^{q+1} - 1 \end{aligned}$$

\# using assumed identity for geometric series

This form also agrees with $T(3q + r)$ when $q = 0$, so I'll conjecture $c(q) = 2^{q+1} - 1$

- 2 marks for 2 successful substitutions
- 0.5 marks for successful derivation with summation
- 0.5 marks for replacing the summation with closed form

(b) **[6 marks]** Let $r \in \{0, 1, 2\}$, and let function c be your conjecture for a closed form in part (a). Use induction on q to prove $\forall q \in \mathbb{N}, c(3q + r) = T(3q + r)$.

If you did not find a sucessful conjecture for c in part (a), for up to 4/6 marks you may show that $\forall q \in \mathbb{N}, T(3q + r) \geq 2^q$. If you choose this option we will **not** grade any attempt to earn the full 6/6.

sample solution: (with suitable closed form) Define $P(q) : T(3q + r) = c(3q + r)$. I will prove $\forall q \in \mathbb{N}, P(q)$ by simple induction on q.

base case(s): $c(3(0) + r) = 2^1 - 1 = 1 = T(r)$, since $r < 3$. So $T(0)$ holds.

inductive step: Let $q \in \mathbb{N}$. Assume $P(q)$, then

$$\begin{aligned} T(3(q+1)+r) &= 1 + 2T(3q+r) \qquad \text{\# by definition, since } 3(q+1) \geq 3 \\ &= 1 + 2c(3q+r) = 1 + 2(2^{q+1} - 1) \qquad \text{\# by IH} \\ &= 1 + 2^{q+2} - 2 = 2^{(q+1)+1} - 1 = c(2(q+1)+r) \end{aligned}$$

So $P(n + 1)$ follows. ■

- 1 mark for defining, or consistently re-stating, predicate
- 1 mark for verifying base case
- 2 mark for introducing q and IH
- 1 mark for using the IH, and saying where you use it
- 1 mark for reaching conclusion

weaker claim: Define $P(q) : T(3q + r) \geq 2^q$. I will prove $\forall q \in \mathbb{N}, P(q)$ by simple induction on q.

base case: By the definition of $T(3(0) + r) = 1 \geq 1 = 2^0$, so $P(0)$ holds.

inductive step: Let $q \in \mathbb{N}$ and assume $P(q)$. I will show that $P(q + 1)$ follows:

$$\begin{aligned} T(3(q+1)+r) &= 1 + 2T(3q+r) \qquad \# \text{ by definition, since } 3(q+1) \geq 3 \\ &\geq 1 + 2 \times 2^q \qquad \# \text{ by } P(q) \\ &= 1 + 2^{q+1} \geq 2^{q+1} \end{aligned}$$

So $P(q + 1)$ follows ■

- 1 mark for defining, or consistently re-stating, predicate
- 1 mark for verifying base case
- 1 mark for introducing q and IH
- 1 mark for using the IH and reaching conclusion

(c) **[4 marks]** Prove that $\forall n \in \mathbb{N}^+, T(n) - T(n-1) \geq 0$. In other words, prove that T is nondecreasing on $\mathbb{N}$. You may assume, as a consequence of the Quotient/Remainder Theorem, that $\forall n \in \mathbb{N}, \exists q, r \in \mathbb{N}, n = 3q + r \wedge 3 > r$.

sample solution: (without induction) Let $n \in \mathbb{N}$ and let $q, r \in \mathbb{N}, n = 3q + r \wedge 3 > r$. There are two cases to consider:

case $r = 0$: Then $n = 3q + 0$ and $n - 1 = 3(q - 1) + 2$, so

$$\begin{aligned} T(n) - T(n-1) &= T(3q+0) + T(3(q-1)+2) = c(3q+0) + c(3(q-1)+2 \qquad \# \text{ by Q1(b)} \\ &= 2^{q+1} - 1 - 2^{q-1+1} + 1 = 2^{q+1} - 2^q = 2^q \geq 0 \end{aligned}$$

case $r \in \{1, 2\}$: Then $n = 3q + r$ and $n - 1 = 3q + r - 1$, where $3 > r - 1 \geq 0$, so

$$\begin{aligned} T(n) - T(n-1) &= T(3q+r) - T(3q+r-1) = c(3q+r) - c(3q+r-1) \qquad \# \text{ by Q1(b)} \\ &= 2^q - 1 - 2^q + 1 = 0 \geq 0 \end{aligned}$$

In both possible cases $T(n) - T(n - 1) \geq 0$ ■

- 1 mark for introducing n in terms of q r
- 1 mark for breaking into cases
- 1 mark for applying closed form
- 1 mark for getting inequality

alternative sample solution: (with induction) Define $P(n) : T(n) - T(n - 1) \geq 0$. I will prove $\forall n \in \mathbb{N}^+, P(n)$ by complete induction

inductive step: Let $n \in \mathbb{N}^+$ and assume $\bigwedge_{i=1}^{i=n-1} P(i)$. I will show that $P(n)$ follows:

case $n \geq 4$:

$$\begin{aligned} T(n) - T(n-1) &= 1 + 2T(n-3) - 1 - 2T(n-4) \qquad \# \text{ since } n-1 \geq 3 \\ &= 2(T(n-3) - T(n-4)) \geq 0 \qquad \# \text{ by } P(n-3),\ 1 \leq n-3 < n \end{aligned}$$

case $1 \leq n \leq 3$: $T(1) - T(0) = 1 - 1 = 0 \geq 0$, and $T(2) - T(1) = 1 - 1 = 0 \geq 0$, and $T(3) - T(2) = 3 - 1 = 2 \geq 0$, so $P(1), P(2)$, and $P(3)$ hold.

In all possible cases, $P(n)$ follows ■

- 1 mark for predicate
- 1 mark for three base cases
- 1 mark for assumptions
- 1 mark for inductive step

2. **[5 marks]** ($\approx$ 15 minutes)

Read over function **pal** below:

```
1  def pal(a_list: list) -> list:
2      """
3      Return a palindrome list based on a_list.
4  
5      Precondition: a_list is a Python list.
6  
7      Postcondition: Returns a_list[ : ] + a_list[ : : -1], where a_list[ : : -1]
8                     is just a_list[ : ] in reverse order.
9      """
10     if len(a_list) < 1:
11         return a_list[ : ]
12     else:
13         return a_list[0] + pal(a_list[1 : ]) + a_list[0]
```

Use induction on the size of **a_list** to prove that **pal's** precondition, plus execution, implies its postcondition. Assume that the reverse-stride slice [1, 2, 3, 4][: : -1] returns [4, 3, 2, 1], that is the reverse of the original list.

sample solution: Define $P(n)$: For any list a_list with len(a_list) $= n$, pal(a_list) returns a_list[:] + a_list[: : -1]. I will prove $\forall n \in \mathbb{N}, P(n)$ by simple induction on n.

base case: By lines 9&10 pal([ ]) returns [ ] = [ ] + [: : -1], so $P(0)$ holds.

inductive step: Let $n \in \mathbb{N}$. Assume $P(n)$. Let a_list be a list with len(a_list)$= n+1$. Then by lines 11&12 pal(a_list) returns:

`a_list[0] + pal(a_list[1 : ]) + a_list[0]`

which, by the inductive hypothesis, since len(a_list[1 :])$= n$, is:

`a_list[0] + a_list[1 : ] + a_list[1 : : -1] + a_list[0]`

... which is `a_list[ : ] + a_list[ : : -1]`. So $P(n+1)$ follows.

- 1 mark for defining, or repeatedly re-stating, predicate
- 2 marks for introducing n and the inductive hypothesis
- 1 mark for using the inductive hypothesis, and making it clear that they have used it
- 1 mark for reaching conclusion