UNIVERSITY OF TORONTO

Faculty of Arts and Science

term test #1, Version 2

CSC236F

Date: Friday October 5, 11:10–12:00pm or 12:10–1:00pm

Duration: 50 minutes

Instructor(s): Danny Heap

Examination Aids: pencils, pens, erasers, drinks, snacks

---

first and last names:

utorid:

student number:

Please read the following guidelines carefully!

- **Please write your name, utorid, and student number on the front of this exam.**
- This examination has **3** questions. There are a total of **4 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely.
- You will receive 20% of the marks for any question you leave blank or indicate "I cannot answer this question."

Take a deep breath.
This is your chance to show us
How much you've learned.

We **WANT** to give you the credit

Good luck!

1. **[7 marks]** ($\approx$ 10 minutes)

Define $f(n)$ by:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ 9 & \text{if } n = 2 \\ f(n-1) + 3f(n-2) + 9f(n-3) & \text{if } n > 2 \end{cases}$$

Define $P(n) : f(n) = 3^n$. Use complete induction to prove $\forall n \in \mathbb{N}, P(n)$. Be sure to introduce names, hypothesis, and all necessary cases. Also be sure to indicate when you use an inductive hypothesis, and why you are justified in using it.

sample solution: Proof by complete induction that $\forall n \in \mathbb{N}, f(n) = 3^n$.

inductive step: Let $n \in \mathbb{N}$. Assume $\bigwedge_{i=0}^{i=n-1} f(i) = 3^i$. I will prove that $P(n)$ follows, that is that $f(n) = 3^n$.

case $n \geq 3$: Then

$$\begin{aligned} f(n) &= f(n-1) + 3f(n-2) + 9f(n-3) \qquad \text{\# by definition, since } n > 2 \\ &= 3^{n-1} + 3(3^{n-2}) + 9(3^{n-3}) \\ &\qquad \text{\# by } P(n-1), P(n-2), \text{ and } P(n-3), n > n-1 > n-2 > n-3 \geq 0 \\ &= 3^{n-1} + 3^{n-1} + 3^{n-1} = 3^n \end{aligned}$$

So $P(n)$ follows in this case.

base cases $n < 3$: By definition, $f(0) = 1 = 3^1$, $f(1) = 3 = 3^1$, and $f(2) = 9 = 3^2$, so $P(n)$ holds in these cases.

So $P(n)$ holds in all possible cases.

2. **[7 marks]** ($\approx$ 20 minutes) Use contradiction and the Principle of Well Ordering to prove that there are no positive integers x, y, z, w such that $x^4 + 3y^4 + 9z^4 = 27w^4$. Be sure to make it clear when you introduce any assumption(s), where you use the Principle of Well Ordering, and where you think you have derived a contradiction. You may assume, without proof, that $\forall p, k \in \mathbb{N}$, if p is prime and $p \mid k^4$, then $p \mid k$.

sample solution: Proof by contradiction and well-ordering that $\forall x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 \neq 27w^4$.

Assume, for the sake of contradiction assume the negation, that $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$. By this assumption I can construct a non-empty set $X = \{x \in \mathbb{N}^+ \mid \exists y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4\}$. Since X is a non-empty subset of $\mathbb{N}$, it has a smallest element. Let $x_0 \in \mathbb{N}^+$ be the smallest element of X, with corresponding $y_0, z_0, w_0 \in \mathbb{N}^+$ such that $x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4$. Then

$$\begin{aligned}
x_0^4 + 3y_0^4 + 9z_0^4 = 27w_0^4 &\Rightarrow x_0^4 = 27w_0^4 - (3y_0^4 + 9z_0^4) \\
&\Rightarrow 3 \mid x_0^4 \Rightarrow 3 \mid x_0 \quad \text{\# since 3 divides RHS and allowed assumption} \\
\text{Let } x_1 \in \mathbb{N}^+, 3x_1 = x_0 &\Rightarrow 81x_1^4 = 27w_0^4 - (3y_0^4 + 9z_0^4) \\
&\Rightarrow 27x_1^4 = 9w_0^4 - (y_0^4 + 3z_0^4) \Rightarrow y_0^4 = 9w_0^4 - (3z_0^4 + 27x_1^4) \quad \text{\# divide by 3} \\
&\Rightarrow 3 \mid y_0^4 \Rightarrow 3 \mid y_0 \quad \text{\# since 3 divides RHS and allowed assumption} \\
\text{Let } y_1 \in \mathbb{N}^+, 3y_1 = y_0 &\Rightarrow 81y_1^4 = 9w_0^4 - (3z_0^4 + 27x_1^4) \\
&\Rightarrow 27y_1^4 = 3w_0^4 - (z_0^4 + 9x_1^4) \Rightarrow z_0^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \quad \text{\# divide by 3} \\
&\Rightarrow 3 \mid z_0^4 \Rightarrow 3 \mid z_0 \quad \text{\# since 3 divides RHS and allowed assumption} \\
\text{Let } z_1 \in \mathbb{N}^+, 3z_1 = z_0 &\Rightarrow 81z_1^4 = 3w_0^4 - (9x_1^4 + 27y_1^4) \\
&\Rightarrow 27z_1^4 = w_0^4 - (3x_1^4 + 9y_1^4) \Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = w_0^4 \quad \text{\# divide by 3} \\
&\Rightarrow 3 \mid w_0^4 \Rightarrow 3 \mid w_0 \quad \text{\# since 3 divides LHS and allowed assumption} \\
\text{Let } w_1 \in \mathbb{N}^+, 3w_1 = w_0 &\Rightarrow 3x_1^4 + 9y_1^4 + 27z_1^4 = 81w_1^4 \\
&\Rightarrow x_1^4 + 3y_1^4 + 9z_1^4 = 27w_1^4 \quad \text{\# divide by 3}
\end{aligned}$$

So $x_1 \in X$ and $x_0 = 3x_1 > x_1$ $\longrightarrow\!\longleftarrow$ Contradiction, x_0 is the smallest element of X. Since assuming $\exists x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 = 27w^4$ yields a contradiction, that assumption is false and:

$$\forall x, y, z, w \in \mathbb{N}^+, x^4 + 3y^4 + 9z^4 \neq 27w^4$$

3. **[7 marks]** ($\approx$ 20 minutes) Define $\mathcal{T}$ as the smallest set such that:

   (a) $() \in \mathcal{T}$

   (b) If $t_1, t_2 \in \mathcal{T}$, then $(t_1 t_2) \in \mathcal{T}$

   Some examples of elements of $\mathcal{T}$ are $()$, $(()())$, and $((()())())$. For $t \in \mathcal{T}$, define $\text{left}(t)$ as the number of (characters in t. Define:

   $$P(t) : \text{left}(t) \text{ is odd.}$$

   Use structural induction to prove $\forall t \in \mathcal{T}, P(t)$. Be sure to indicate the cases you present, when you introduce names, where you introduce assumptions, and when you have derived a conclusion.

   sample solution: Proof by structural induction that $\forall t \in \mathcal{T},\ \text{left}(t)$ is odd.

   basis: Let $t = ()$. Then $\text{left}(t) = 1 = 2 \times 0 + 1$, an odd number. So $P(t)$ holds.

   inductive step: Let $t_1, t_2 \in \mathcal{T}$ and assume $P(t_1)$ and $P(t_2)$. Let $j, k \in \mathbb{Z}$ such that $\text{left}(t_1) = 2j+1$ and $\text{left}(t_2) = 2k+1$. Let $i \in \mathbb{Z}, i = j + k + 1$. I will show that $P((t_1 t_2))$ follows, that is $\text{left}((t_1 t_2)) = 2i + 1$, an odd number.

   $$\begin{aligned} \text{left}((t_1 t_2)) &= 1 + \text{left}(t_1) + \text{left}(t_2) && \text{\#exactly one new (added} \\ &= 1 + (2j+1) + (2k+1) && \text{\# by } P(t_1) \wedge P(t_2) \\ &= 2(j+k+1) + 1 = 2i + 1 \end{aligned}$$

   So $P((t_1 t_2))$ follows in this case.