CSC236 tutorial exercises, Week #2
sample solutions

Solve question 1, then prove claims 2–4 using Mathematical Induction (AKA Simple Induction).

1. Define $P(n)$ as:

$$\sum_{i=0}^{i=n} 2^i = 2^{n+1}$$

   (a) Prove that $P(115)$ implies $P(116)$.

   proof: Assume $P(115)$, that is $\sum_{i=0}^{i=115} 2^i = 2^{116}$. I must now show that $P(116)$ follows. Notice that

$$\begin{aligned} \sum_{i=0}^{i=116} 2^i &= \left[\sum_{i=0}^{i=115} 2^i\right] + 2^{116} && \text{\# regrouping} \\ &= 2^{116} + 2^{116} && \text{\# by } P(115) \\ &= 2^{117} \quad \blacksquare \end{aligned}$$

   It is also possible to note that $P(115)$ is false, and an implication with a false hypothesis is always true (vacuous truth).

   (b) Is $P(n)$ true for every natural number n? Explain why, or why not.

   solution: $P(n)$ is false for **every** natural number n. Because of this it is impossible to verify a base case, so the correct induction step (see above) does not establish a proof.

2. $\forall n \in \mathbb{N}$, $8^n - 1$ is a multiple of 7.

   proof by simple induction: For convenience define $H(n) : \exists k \in \mathbb{N}, 8^n - 1 = 7k$. I will prove $\forall n \in \mathbb{N}, H(n)$.

   base case: $8^0 - 1 = 0 = 7 \times 0$, which verifies $H(0)$.

   inductive step: Let $n \in \mathbb{N}$ and assume $H(n)$, let k be such that $8^n - 1 = 7k$. Let $k' = 8k + 1$. I will show that $8^{n+1} - 1 = 7k'$.

$$\begin{aligned} 8^{n+1} - 1 &= 8(8^n - 1) + 7 \\ &= 8(7k) + 7 && \text{\# by } H(n) \\ &= 7(8k+1) = 7k' \quad \blacksquare && \text{\# by choice of } k' \end{aligned}$$

3. $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$, the units digit of 7^n is the same as the units digit of 3^m.

   proof by simple induction: For convenience define $H(n) : \exists m \in \mathbb{N}, 7^n \equiv 3^m \bmod 10$. I must prove $\forall n \in \mathbb{N}, H(n)$.

base case: $7^0 = 1 = 3^0$, which verifies $H(0)$.

inductive step: Let $n \in \mathbb{N}$. Assume $H(n)$, and let m b e such that

$$7^n \equiv 3^m \bmod 10$$

Let $m' = m + 3$. I will show $C(n)$ follows, that is

$$7^{n+1} \equiv 3^{m+3} \bmod 10$$

Note that $3^3 \equiv 7 \bmod 10$, so

$$\begin{aligned} 7 \times 7^n &\equiv 3^3 \times 7^n \bmod 10 && \text{\# by Example 2.18, CSC165 notes} \\ &\equiv 3^3 \times 3^m \bmod 10 && \text{\# by } H(n) \text{ and Example 2.18 again} \\ &\equiv 3^{m+3} \bmod 10 \quad \blacksquare \end{aligned}$$

4. $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq m \Rightarrow 4^n \geq 5n^4 + 6$

proof by simple induction: For convenience define $H(n) : 4^n \geq 5n^4 + 6$. Let $m = 7$. I will prove $\forall n \in \mathbb{N}, n \geq 7 \Rightarrow H(n)$.

base case: $4^7 - 5(7^4) - 6 = 4373$, which verifies $H(7)$.

induction step: Let $n \in \mathbb{N} - \{0, 1, 2, 3, 4, 5, 6\}$. Assume $H(n)$, that is $4^n \geq 5n^4 + 6$. I will show that $C(n)$ follows, that is $4^{n+1} \geq 5(n+1)^4 + 6$.

$$\begin{aligned} 4^{n+1} &= 4 \times 4^n \geq 4(5n^4 + 6) = 20n^4 + 24 && \text{\# by } H(n) \\ &=\geq 5(n^4 + n^4 + n^4 + n^4) + 6 \geq 5(n^4 + 7n^3 + 49n^2 + 343n) + 6 && \text{\# since } n \geq 7 \\ &\geq 5(n^4 + 4n^3 + 6n^2 + 4n + 339n) + 6 && \text{\# since } 7 \geq 4 \wedge 49 \geq 6 \\ &\geq 5(n^4 + 4n^3 + 6n^2 + 4n + 1) + 6 && \text{\# since } n \geq 7 \geq 1/339 \\ &= 5(n+1)^4 + 6 \quad \blacksquare \end{aligned}$$