1. (10 pts) Consider the following recurrence relation:

$$T(n) = \begin{cases} 3 & n = 2 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) * T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) & n > 2 \end{cases}$$

Use repeated substitution (aka unwinding) to make a conjecture of a closed-form expression for $T(n)$ in the special case where n is a power of 2 (i.e.,$\exists k > 0 \in \mathbb{N}, n = 2^k$). Then, prove your conjecture is true for n of the form 2^k.

Sample Solution.

When $n > 2$

$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) * T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = T\left(\frac{n}{2}\right) * T\left(\frac{n}{2}\right) = T^2\left(\frac{n}{2}\right)$ # since $n = 2^k$

$= T^4\left(\frac{n}{4}\right)$

$= T^8\left(\frac{n}{8}\right)$

…

$= T^{2^k}\left(\frac{n}{2^k}\right)$

$= T^{\frac{n}{2}}(2)$ # when $2^k = \frac{n}{2}$

$= 3^{\frac{n}{2}}$

Proof by simple induction on k.

Basis step. $k = 1, n = 2^1 = 2.$ $T(2) = 3 = 3^{\frac{2}{2}} = 3.$

Inductive step. Assume $P(k)$ holds for an arbitrary $k > 0 \in N$. That means

$T(2^k) = T^2\left(\frac{2^k}{2}\right) = T^2(2^{k-1})$=$3^{(2^{k-1})}$

By using IH, we must show $T(2^{k+1}) = 3^{(2^k)}$.

$T(2^{k+1}) = T^2\left(\frac{2^{k+1}}{2}\right)$

$= T^2(2^k)$

$= (3^{(2^{k-1})})^2$ # by IH.

$= 3^{(2^{k-1})*2} = 3^{(2^k)}$ □

2. (8 pts) Assume we know that when $n = 2^{(2^k)}$ for some $k \in \mathbb{N}$, $S(n) = \lg\lg n + 3$. Show that $S(n)$ is in $\Omega(\lg\lg n)$ for all $n > 1 \in \mathbb{N}$, not just special cases. **Hint:** you may use

i. S is monotonic non-decreasing

ii. $\forall\, n > 1 \in \mathbb{N}, \exists k \in \mathbb{N}$ such that $\sqrt{2^{(2^k)}} \le n \le 2^{(2^k)}$

iii. $\sqrt{2^{(2^k)}} = 2^{(2^{k-1})}$

iv. Since $n = 2^{(2^k)}$, $2^k = \lg n$ and $k = \lg\lg n$

Note that we want to show $S(n)$ <u>for all $n > 1 \in \mathbb{N}$</u> is in $\Omega(\lg\lg n)$.

Sample Solution.

Assume $\sqrt{2^{2^k}} \le n \le 2^{2^k}$

$S(n) \ge S\left(\sqrt{2^{2^k}}\right)$ # since S is non-decreasing

$S(n) \ge \lg\lg\sqrt{2^{2^k}} + 3$

$S(n) \ge \lg\lg 2^{(2^{k-1})} + 3$

$S(n) \ge k - 1 + 3$

$S(n) \ge k$

$S(n) \ge \lg\lg n$ # since $k = \lg\lg n$

□

Note. *This question may take more time than the number of points assigned suggest.*

3.

a) (4 pts) Consider the following algorithm, and prove if the loop iterates at least c times, the following loop invariant holds at end of the c-th iteration.

$$LI(i_c, sum_c): 0 \le i_c \le \frac{n}{2}, i_c \in \mathbb{N} \text{ and } sum_c = \sum_{j=0}^{i_c-1} A[2j]$$

Note that sum of an empty list is zero, *i.e.*, $\sum_{j=0}^{-1} A[2j] = 0$.

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Algorithm avg(A)
pre-: A is a list of real numbers, its index starts from 0 and
its size, n, is 2k, ∃k > 0 ∈ ℕ
post-: return the average of numbers in positions divisible by 2
i = 0
sum = 0
m = length(A)/2
while i < m
    sum = sum + A[2 * i]
    i = i + 1
a = sum / i
return a
```

b) (1 pts) Partial correctness: use the loop invariant above and prove the algorithm is correct, assuming it terminates.

sample solution

a)

Proof by simple induction.

Basis step.

$P(0)$ holds since if the loop iterates at least 0 times, i.e. before entering the loop:

$$0 \le i_0 = 0 \le \frac{n}{2} \quad , i_0 = 0 \in \mathbb{N} \text{ and}$$

$$Sum_c = 0 = \sum_{j=0}^{i_0-1} A[2j] = \sum_{j=0}^{-1} A[2j] = 0$$

Inductive step.

Assume $P(k)$ holds, *i.e.*, if the loop iterates at least k times, then $i_k \in \mathbb{N}$, $0 \le i_k \le \frac{n}{2}$ *and* $Sum_k = \sum_{j=0}^{i_k-1} A[2j]$. We must show $P(k) \rightarrow P(k+1)$.

Case 1: if the loop does not iterate $k+1$ times, $P(k+1)$ is vacuously true.
Case 2: If the loop iterates at least $k+1$ times,

$0 \le i_k < m = \frac{n}{2}$ # by Line 5

Then,

$Sum_{k+1} = Sum_k + A[2 * i_k]$ # at Line 6

$= \sum_{j=0}^{i_k - 1} A[2j] + A[2 * i_k]$# by IH

$= \sum_{j=0}^{i_{k+1} - 1} A[2j]$ # since $i_k = i_{k+1} - 1$

Also,

$i_{k+1} = i_k + 1$ # Line 7

$0 \le i_{k+1} \le \frac{n}{2}$ # since $0 \le i_k < \frac{n}{2}$ and $i_K \in \mathbb{N}$

Also,

$i_{k+1} \in \mathbb{N}$ # since $i_K \in \mathbb{N}$, and $i_{k+1} = i_k + 1$,

This completes the inductive step, as $P(k+1)$ holds.

Hence, if the loop iterates at least c times, the following loop invariant holds at end of the c-th iteration

□

b)

Sample solution.
Since the loop terminates and by LI $0 \le i_c \le \frac{n}{2}$ at end of iteration c, then $i_c = m = \frac{n}{2}$ when the loop exits. Also, by LI, when the loop exits:
$Sum_{\frac{n}{2}} = \sum_{j=0}^{\frac{n}{2}-1} A[2j]$ which is sum of elements of A at positions divisible by 2, up to and including position $n-2$. The number of elements at positions divisible by 2 is $\frac{n}{2} = i_{\frac{n}{2}}$. The program returns $Sum_{\frac{n}{2}}/i_{\frac{n}{2}}$ which is the average by definition.

Hence, $precondtions \rightarrow postconditions$

□