PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term test #2, L0201

CSC 236H1
Duration — 50 minutes

PLEASE HAND IN

Last Name: ____

First Name: ____

Do **not** *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 8 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on.

Good Luck!

Question 1. [10 MARKS]

Consider the function:

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ \left[f(\lfloor \frac{n}{2} \rfloor)\right]^2 + 2f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 0 \end{cases}$$

Prove that $f(n)$ is divisible by 10 for most natural numbers. As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

Question 2. [10 MARKS]

Recall the function from the previous question:

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ [f(\lfloor n/2 \rfloor)]^2 + 2f(\lfloor n/2 \rfloor) & \text{if } n > 0 \end{cases}$$

Define $P(n)$: "$\forall m \in \mathbb{N}, m \leq n \Rightarrow f(m) \leq f(n)$". Use Complete Induction to prove that $P(n)$ is true for all natural numbers n. As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

For convenience, you may assume that for natural numbers m, n, if $m \leq n$ then $\lfloor \frac{m}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$, and that for all natural numbers n, $f(n) \geq 1$. Neither of these are hard to prove, but they distract from the main point.

Question 3. [10 MARKS]

Consider the recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + n & \text{if } n > 1 \end{cases}$$

Make an educated guess at a closed form for $T(2^k)$ when k is a natural number. State your guess, then prove your guess is correct, using Simple Induction on k. If you are unable to come up with a suitable guess, you will get substantial part marks for proving that $T(2^k) \leq 2^{k+1}$ for all natural numbers k.

This page is left (nearly) blank to accommodate work that wouldn't fit elsewhere.

1: ______/10

2: ______/10

3: ______/10

TOTAL: ______/30