

PLEASE HAND IN

UNIVERSITY OF TORONTO  
Faculty of Arts and Science

Term test #2

CSC 236H1  
Duration — 50 minutes

PLEASE HAND IN

Last Name: \_\_\_\_\_

First Name: \_\_\_\_\_

*Do not turn this page until you have received the signal to start.*  
(In the meantime, please fill out the identification section above,  
and read the instructions below.)

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This test consists of 3 questions on 4 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write “I cannot answer this question,” on.

*Good Luck!*

**Question 1.** [10 MARKS]

Consider the function:

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ [f(\lfloor \frac{n}{2} \rfloor)]^2 + 2f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 0 \end{cases}$$

Prove that  $f(n)$  is divisible by 10 for most natural numbers. As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

**Sample solution:** After some experimentation, it seems that the claim  $P(n) : f(n) = 10k$ , for some  $k \in \mathbb{N}$ , is true for all natural numbers  $n$  larger than 1.

**Proof, by Complete Induction:** Assume  $n \in \mathbb{N}$  is no smaller than 2, and that  $P(i)$  is true for  $2 \leq i < n$  (Inductive Hypothesis, IH). There are two cases to consider:

**Case  $n \geq 4$ :** In this case we have

$$\begin{aligned} f(n) &= \left[ f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \right]^2 + 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) && \# \text{ by definition of } f(n), n > 1 \\ &= (10k)^2 + 2(10k), k \in \mathbb{N} && \# \text{ by IH for } 2 \leq \left\lfloor \frac{n}{2} \right\rfloor < n \\ &= 10(10k^2 + 2k), 10k^2 + 2k \in \mathbb{N} && \# \text{ multiples and sum of natural numbers} \end{aligned}$$

So  $P(n)$  follows in this case

**Base case  $n \in \{2, 3\}$ :** Since  $f(1) = 8$  and  $n > 0$ , we have:

$$\begin{aligned} f(n) &= \left[ f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \right]^2 + 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = f(1)^2 + 2f(1) = 80 && \# \text{ since } \left\lfloor \frac{n}{2} \right\rfloor = 1 \\ &= 8 \times 10, 8 \in \mathbb{N} \end{aligned}$$

so  $P(n)$  follows in this case.

**I conclude, by Complete Induction,**  $\forall n \in \mathbb{N}, n \geq 2 \Rightarrow P(n)$ .

**Question 2.** [10 MARKS]

Recall the function from the previous question:

$$f(n) = \begin{cases} 2 & \text{if } n = 0 \\ [f(\lfloor n/2 \rfloor)]^2 + 2f(\lfloor n/2 \rfloor) & \text{if } n > 0 \end{cases}$$

Define  $P(n)$ : " $\forall m \in \mathbb{N}, m \leq n \Rightarrow f(m) \leq f(n)$ ". Use Complete Induction to prove that  $P(n)$  is true for all natural numbers  $n$ . As usual, label your Inductive Hypothesis (IH), and when you use your IH, mention which numbers you're using it for and why this is valid.

For convenience, you may assume that for natural numbers  $m, n$ , if  $m \leq n$  then  $\lfloor \frac{m}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$ , and that for all natural numbers  $n$ ,  $f(n) \geq 1$ . Neither of these are hard to prove, but they distract from the main point.

**Sample solution:** There are a few ways to do this, but I'll take the approach that decomposes  $f(m)$  and  $f(n)$  and uses the assumption that their components are monotonic.

**Proof, by Complete Induction:** Assume that  $n \in \mathbb{N}$ , that  $m$  is a natural number with  $0 \leq m \leq n$ , and that  $P(i)$  is true for  $0 \leq i < n$  (Inductive Hypothesis, IH). I must now show that  $P(n)$  follows.

**Case  $n > 0$**  There are two subcases:

**Subcase  $m = 0$ :** In this case we have

$$\begin{aligned} f(m) &= f(0) \leq f(\lfloor \frac{n}{2} \rfloor) && \# \text{ by IH for } 0 \leq \lfloor \frac{n}{2} \rfloor < n \\ &< 2f(\lfloor \frac{n}{2} \rfloor) && \# \text{ since } f(\lfloor \frac{n}{2} \rfloor) \geq f(0) = 2 > 0 \\ &\leq f(\lfloor \frac{n}{2} \rfloor)^2 + 2f(\lfloor \frac{n}{2} \rfloor) && \# \text{ since } f(\lfloor \frac{n}{2} \rfloor)^2 > 0 \\ &= f(n) && \# \text{ by definition of } f(n) \end{aligned}$$

So  $P(n)$  follows in this case.

**Subcase  $m > 0$ :** In this case, we have

$$\begin{aligned} f(m) &= f(\lfloor \frac{m}{2} \rfloor)^2 + 2f(\lfloor \frac{m}{2} \rfloor) && \# \text{ by definition of } f(m), m > 0 \\ &\leq f(\lfloor \frac{n}{2} \rfloor)^2 + 2f(\lfloor \frac{n}{2} \rfloor) && \# \text{ by IH for } \lfloor \frac{n}{2} \rfloor, \text{ since } 0 \leq \lfloor \frac{m}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor < n \\ &= f(n) && \# \text{ by definition of } f(n), n \geq m > 0 \end{aligned}$$

So  $P(n)$  follows in this case.

**Base case,  $n = 0$ :** Since  $m$  is assumed to be a natural number no larger than  $n$ , the only possibility we have is  $m = n = 0$ , so  $f(m) = f(0) \leq f(0) = f(n)$ , so  $P(n)$  follows in this case.

**I conclude, by Complete Induction,**  $\forall n \in \mathbb{N}, P(n)$ .

**Question 3.** [10 MARKS]

Consider the recurrence:

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + n & \text{if } n > 1 \end{cases}$$

Make an educated guess at a closed form for  $T(2^k)$  when  $k$  is a natural number. State your guess, then prove your guess is correct, using Simple Induction on  $k$ . If you are unable to come up with a suitable guess, you will get substantial part marks for proving that  $T(2^k) \leq 2^{k+1}$  for all natural numbers  $k$ .

**Sample solution:** Repeated substitution, AKA unwinding/unrolling suggests a closed form:

$$\begin{aligned} T(2^k) &= T(2^{k-1}) + 2^k = T(2^{k-2}) + 2^{k-1} + 2^k = T(2^{k-3}) + 2^{k-2} + 2^{k-1} + 2^k \\ &\vdots \\ &= T(2^0) + 2^1 + \cdots + 2^{k-1} + 2^k = 2^0 + 2^1 + \cdots + 2^k \end{aligned}$$

I recognize the sum  $1 + 2 + 2^2 + \cdots + 2^k$  as  $2^{k+1} - 1$ , so I set out to prove  $P(k) : T(2^k) = 2^{k+1} - 1$  is true for all natural numbers  $k$ .

**Proof by Simple Induction:**

**Base case,  $k = 0$ :** I verify that  $T(0) = 1 = 2^1 - 1 = 2^{0+1} - 1$ . So  $P(0)$  is true.

**Inductive step:** Assume that  $k \in \mathbb{N}$  and  $P(k)$  (Inductive Hypothesis, IH). Now I have to show  $P(k+1)$ .

Since  $k \in \mathbb{N}$  we must have  $k+1 > 0$ , so

$$\begin{aligned} T(2^{k+1}) &= T(2^k) + 2^{k+1} && \# \text{ by definition for } T(2^{k+1}), 2^{k+1} > 1 \\ &= 2^{k+1} - 1 + 2^{k+1} && \# \text{ by IH for } k \\ &= 2^{k+2} - 1 \end{aligned}$$

So  $P(k+1)$  follows.

**Conclude, by Simple Induction,  $\forall k \in \mathbb{N}, P(k)$ .**

# 1: \_\_\_\_\_/10

# 2: \_\_\_\_\_/10

# 3: \_\_\_\_\_/10

TOTAL: \_\_\_\_\_/30