1. ( 7 pts) Use simple induction to prove that $3^{n}>n^{3}+n$ for all natural numbers $n>3$. You may use the fact that $(n+1)^{3}=n^{3}+3 n^{2}+3 n+1$. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

A, weight 1 : verify base case
$B$, weight 1 : introduce generic element of domain
C, weight 1: state inductive hypothesis
D, weight 1: reference IH when used
E, weight 3: derive inductive conclusion
sample solution: Define $P(n): 3^{n}>n^{3}+n$. Proof by simple induction that $\forall n \in$ $\mathbb{N} \backslash\{0,1,2,3\}, P(n)$.
base case: $3^{4}=81>68=4^{3}+4$, so $P(4)$ holds.
inductive step: Let $n$ be an arbitrary natural number greater than 3. Assume $H(n)$, that is $3^{n}>n^{3}+n$.
derive inductive conclusion $C(n)$ : that is, $3^{n+1}>(n+1)^{3}+(n+1)$.

$$
\begin{aligned}
3^{n+1} & =3 \times 3^{n}>3\left(n^{3}+n\right) \quad \# \text { by } H(n) \\
& =3 n^{3}+3 n=n^{3}+n n^{2}+n^{2} n+3 n \quad \text { \# expanding } \\
& >n^{3}+3 n^{2}+4 n+2 \quad \# n \geq 6>3, n^{2} \geq 36>4, n \geq 6>2 / 3 \\
& =(n+1)^{3}+(n+1) \quad \# \text { re-writing }
\end{aligned}
$$

So $C(n)$
2. (12 pts) Read the definition of function $f$ :

$$
f(n)= \begin{cases}1 & \text { if } n=0 \\ 1 & \text { if } n=1 \\ 2 & \text { if } n=2 \\ (f(\lfloor n / 3\rfloor))^{2}+f(\lfloor n / 3\rfloor) & \text { if } n>2\end{cases}
$$

Use complete induction to prove that for every natural number $n$ larger than 1 , $f(n)$ is even. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, where the inductive hypothesis is used, and why it is valid to use it there.

A, weight 4: verify base cases
$B$, weight 1 : introduce generic element of domain
C, weight 2: state inductive hypothesis
D, weight 1: reference IH when used
E, weight 1: jusify that IH applies to value(s)
D, weight 3: derive inductive conclusion
sample solution: Define $P(n): f(n)$ is a multiple of 2 . Proof by complete induction that $\forall n \in \mathbb{N} \backslash\{0,1\}, P(n)$.
inductive step: Let $n \in \mathbb{N} \backslash\{0,1\}$. Assume $H(n): \forall i \in \mathbb{N}$ if $2 \leq i<n$ then $f(i)$ is a multiple of 2.
must show inductive hypothesis $C(n): f(n)$ is a multiple of 2 .
Base case, $n=2$ : Then $f(n)=2$, from definition, and 2 is notoriously a multiple of 2 .
Base case, $n \in\{3,4,5\}$ : Then

$$
\begin{aligned}
f(n) & =(f(\lfloor n / 3\rfloor))^{2}+f(\lfloor n / 3\rfloor) \quad \text { from definition of } f(n), n>2 \\
& =(f(1))^{2}+f(1) \quad \#\lfloor n / 3\rfloor=1 \text { for } n \in\{3,4,5\} \\
& =1+1=2 \quad \text { \# from definition of } f(1), 2 \text { is a multiple of } 2
\end{aligned}
$$

Case $n>5: f(\lfloor n / 3\rfloor)$ is a multiple of 2 , by $H(n)$, since $2 \leq\lfloor n / 3\rfloor<n$ when $n$ is at least 6 . Let $k \in \mathbb{N}$ such that $f(\lfloor n / 3\rfloor)=2 k$

$$
\begin{aligned}
f(n) & =(f(\lfloor n / 3\rfloor))^{2}+f(\lfloor n / 3\rfloor) \quad \text { from definition of } f(n), n>2 \\
& =(2 k)^{2}+2 k=2\left(2 k^{2}+k\right)
\end{aligned}
$$

$$
\# \text { a multiple of } 2 \text { since } 2, k \in \mathbb{N} \text { and } \mathbb{N} \text { is closed under }+, \times
$$

In every possible case $C(n)$ follows
3. ( 10 pts ) Define the set of non-empty binary trees $\mathcal{B} \mathcal{T}^{*}$ (not full binary trees) as the smallest set such that:
a) A solitary node with no edges is an element of $\mathcal{B T ^ { * }}$.
b) If $t_{1}, t_{2} \in \mathcal{B} \mathcal{T}^{*}$ and $n$ is a single node, then the following three trees are also elements of $\mathcal{B} \mathcal{T}^{*}$ :
i. the tree formed with root $n$ and an edge to left subtree $t_{1}$;
ii. the tree formed with root $n$ and an edge to right subtree $t_{2}$;
iii. the tree formed with root $n$, an edge to left subtree $t_{1}$, and an edge to right subtree $t_{2}$.
For $t \in \mathcal{B} \mathcal{T}^{*}$ define $N(t)$ as the number of nodes in $t$, and $E(t)$ as the number of edges in $t$. Use structural induction to prove that for all $t \in \mathcal{B T}^{*}$, $N(t)=E(t)+1$. For full marks your proof must clearly indicate any base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

A, weight 1 : verify basis
B, weight 2: introduce generic elements of the domain
C, weight 2: state inductive hypothesis
D, weight 2: reference inductive hypothesis:
E, weight 2: consider 3 cases
F, weight 1: derive conclusion
sample solution: Define $P(t): N(t)=E(t)+1$. Proof by structural induction that $\forall t \in \mathcal{B} \mathcal{T}^{*}, P(t)$
verify basis: A solitary node $t$ with no edges has $N(t)=1=0+1=E(t)+1$. So the claim holds for the basis.
inductive step: Let $t_{1}, t_{2}$ be arbitrary elements of $\mathcal{B T ^ { * }}$ and $n$ be a single node. Assume $H\left(\left\{t_{1}, t_{2}\right\}\right)$, that is $P\left(t_{1}\right)$ and $P\left(t_{2}\right)$.
must show $C\left(\left\{t_{1}, t_{2}\right\}\right)$ : that is, any tree $t$ in $\mathcal{B} \mathcal{T}^{*}$ formed from root $n$, with subtrees $t_{1}, t_{2}$ satisfies $P(t)$. There are 3 cases to consider:
Case i: If $t$ is a tree formed from root $n$ with left subtree $t_{1}$, then $N(t)=$ $N\left(t_{1}\right)+1$, since $n$ provides one new node. Also $E(t)=E\left(t_{1}\right)+1$, since there is a new edge from $n$ to its subtree. Summing up:

$$
\begin{aligned}
N(t) & =N\left(t_{1}\right)+1=\left(E\left(t_{1}\right)+1\right)+1 \quad \# \text { by } H\left(\left\{t_{1}, t_{2}\right\}\right. \\
& =E(t)+1 \quad \# \text { since } n \text { add one node. }
\end{aligned}
$$

$P(t)$ follows in this case.
Case ii: If $t$ is a tree formed from root $n$ with right subtree $t_{2}$, the argument is the same as Case ii with $t_{1}$ replace by $t_{2}$, and $P(t)$ follows in this case.
Case iii: If $t$ is a tree formed from room $n$ with left subtree $t_{1}$ and right subtree $t_{2}$, then $N(t)$ is the sum of the nodes in the two subtrees, plus one for $n$ itself, or $N(t)=N\left(t_{1}\right)+N\left(t_{2}\right)+1$. $E(t)$ is the sum of the edges in the two subtrees, plus two edges connecting $t$ to each subtree, or $E(t)=E\left(t_{1}\right)+E\left(t_{2}\right)+2$. Summing up:

$$
\begin{aligned}
N(t) & =N\left(t_{1}\right)+N\left(t_{2}\right)+1= \\
& =\left(E\left(t_{1}\right)+1\right)+\left(E\left(t_{2}\right)+1\right)+1 \quad \text { \# by } H\left(\left\{t_{1}, t_{2}\right\}\right) \\
& =\left(E\left(t_{1}\right)+E\left(t_{2}\right)+2\right)+1=E(t)+1
\end{aligned}
$$

$P(t)$ follows in this case.
In every possible case $P(t)$ follows, that is $C\left(\left\{t_{1}, t_{2}\right\}\right)$

