mentors.... list RSN
instructional support says... today!

CSC236 fall 2018

divide and conquer : as a design tool

recursive correctness

Danny Heap
heap@cs.toronto.edu / BA4270 (behind elevators)
http://www.teach.cs.toronto.edu/~heap/236/F18/
416-978-5899

Using Introduction to the Theory of Computation, Chapter 3

Outline

general D&C case

revisit...

a: number of recursive calls (a1 + a2)
b: number of pieces we divide into
f(n) ~ n^d, cost of dividing and then recombining

Class of algorithms: partition problem into b *roughly* equal subproblems, solve, and recombine:

$$T(n) = \begin{cases} k & \text{if } n \leq b \\ a_1\, T(\lceil n/b \rceil) + a_2\, T(\lfloor n/b \rfloor) + f(n) & \text{if } n > b \end{cases}$$

where $b, k > 0$, $a_1, a_2 \geq 0$, and $a_1 + a_2 > 0$. $f(n)$ is the cost of splitting and recombining.

Master Theorem

(for divide-and-conquer recurrences)

If f from the previous slide has $f \in \theta(n^d)$, then

$$T(n) \in \begin{cases} \theta(n^d) & \text{if } a < b^d \\ \theta(n^d \log_b n) & \text{if } a = b^d \\ \theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

multiply lots of bits

what if they don't fit into a machine instruction?

```
     1101
    ×1011
   -------
     1101     n copies, Theta(n^2)
    1101      n column-wise
   0000       additions, Theta(n^2)
  1101
---------
 10001111
```

divide and recombine

recursively… $2^n = n$ left-shifts, and addition/subtraction are $\Theta(n)$

```
1101 = (1100 + 01) = (11x2^2 + 01)
1011 = (1000 + 11) = (10x2^2 + 11)
```

11	01
×10	11

$$
\begin{aligned}
xy &= (2^{n/2}x_1 + x_0)(2^{n/2}y_1 + y_0) \\
&= 2^n x_1 y_1 + 2^{n/2}(x_1 y_0 + y_1 x_0) + x_0 y_0
\end{aligned}
$$

compare costs

n n-bit additions versus:

1. divide each factor (roughly) in half   b = 2
2. multiply the halves (recursively, if they're too big)   a = 4
3. combine the products with shifts and adds   d = 1

4 > 2^1

what?!? back to Theta(n^2)

Gauss's trick

$$xy = 2^n x_1 y_1 + 2^{n/2} x_1 y_1 + 2^{n/2} \left((x_1 - x_0)(y_0 - y_1) + x_0 y_0 \right) + x_0 y_0$$

Gauss's payoff

lose one multiplication!

1. divide each factor (roughly) in half
2. subtract the halves...
3. multiply the difference and the halves **Gauss-wise**
4. combine the products with shifts and adds

b = 2

d = 1

a = 3

d = 1

3 > 2^1

Theta(n^{log_2 3})

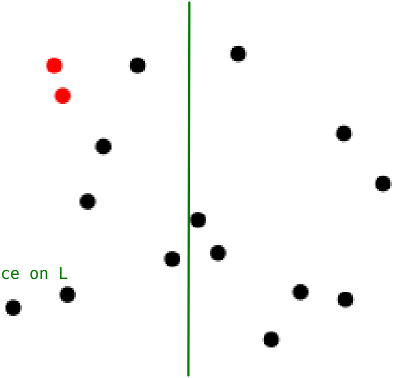
FFT

closest point pairs

see Wikipedia

```
P = [(x0, y0), (x1, y1), ...,  (xn, yn)]

brute force: Theta(n^2)
before recursion, sort into Px and Py: same points, ordered by x, ordered by y
cost: n lg n
```



```
Rx

Ry

Lx

Ly

min_y: min pair distance
on R

min_x: min pair distance on L

delta = min(min_L, min_R)
```

divide-and-conquer v0.1

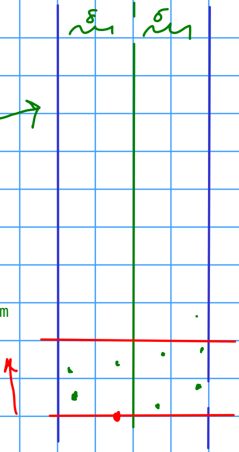
```
b = 2
a = 2
d = ?? 1, it turns out!
```

an $n \lg n$ algorithm

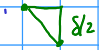
P is a set of points

1. Construct (sort) P_x and P_y before recursion, do it once: n lg n
2. For each recursive call, construct ordered L_x, L_y, R_x, R_y must be in linear time
3. Recursively find closest pairs (l_0, l_1) and (r_0, r_1), with minimum distance δ a = 2
4. V is the vertical line splitting L and R, D is the δ-neighbourhood of V, and D_y is D ordered by y-ordinate must be in linear time
5. Traverse D_y looking for mininum pairs 7 places apart
6. Choose the minimum pair from D_y versus (l_0, l_1) and (r_0, r_1).

Dy



traverse Dy bottom
to top...



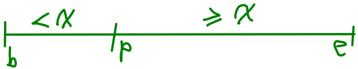
b = 2
a = 2
d = 1

Theta(n lg n)

recursive binary search

A: list, nondecreasing, comparable elements
x: value to search for, must be comparable
b: beginning index of search
e: end index of search

```
def recBinSearch(x, A, b, e) :
  if b == e :
    if x <= A[b] :
      return b
    else :
      return e + 1
  else :
    m = (b + e) // 2 # midpoint
    if x <= A[m] :
      return recBinSearch(x, A, b, m)
    else :
      return recBinSearch(x, A, m+1, e)
```



return position p where x is, or should be inserted.

1. b <= p <= e + 1
2. b < p => A[p-1] < x
3. p < e + 1 => A[p] >= x

conditions, pre- and post-

- x and elements of A are comparable
- e and b are valid indices, $0 \leq b \leq e < len(A)$
- $A[b..e]$ is sorted non-decreasing

RecBinSearch(x, A, b, e) terminates and returns index p

- $b \leq p \leq e + 1$
- $b < p \Rightarrow A[p - 1] < x$
- $p \leq e \Rightarrow x \leq A[p]$

(except for boundaries, returns p so that $A[p - 1] < x \leq A[p]$)

precondition ⇒ termination and postcondition

Proof: induction on $n = e - b + 1$

verify these easily.

Base case, $n = 1$: Terminates because there are no loops or further calls, returns $p = b = e \Leftrightarrow x \leq A[b = p]$ **or** $p = b + 1 = e + 1 \Leftrightarrow x > A[b = p - 1]$, so postcondition satisfied. Notice that the choice forces if-and-only-if.

Induction step: Assume $n > 1$ and that the postcondition is satisfied for inputs of size $1 \leq k < n$ that satisfy the precondition, and the RecBinSearch terminates on such inputs. Call RecBinSearch(A,x,b,e) when $n = e - b + 1 > 1$. Since $b < e$ in this case, the test on line 1 fails, and line 7 executes. **Exercise:** $b \leq m < e$ in this case. There are two cases, according to whether $x \leq A[m]$ or $x > A[m]$.

recursive call's postcondition becomes the IH

Case 1: $x \leq A[m]$

0 <= 0 + 1<= m - b + 1 < e - b + 1 = n
since b <= m < e, by exercise

- Show that IH applies to RBS(x,A,b,m)
- Translate the postcondition to RBS(x,A,b,m) e

These are now our I.H.
1. b <= p <= m+1 # by postcondition
2. b < p ==> A[p-1] < x
3. p <= m ==> A[p] >= x

- Show that RBS(x,A,b,e) satisfies postcondition

1. b <= p # by IH
   p <= m+1 <= e+1 # since m < e, by exercise
2. b<p => A[p-1] < x # by IH
3. p <= e # always true, since p <= m+1 <= e
   => p = m+1 O p <= m # first case NEVER happens
   # since p = m+1 => A[p-1] = A[m] => A[m] < x (contradiction!)
   A[p] >= x # by 3. in IH

Case 2: $x > A[m]$

must show that 1 <= e - m < e - b + 1 = n
since b <= m < e

- Show that IH applies to RBS(x,A,m+1,e)

b

- Translate postcondition to RBS(x,A,m+1,e)

terminates, and
1. m+1 <= p <= e+1
2. m+1 < p ==> A[p-1] < x
3. p <= e ==> A[p] >= x

- Show that RBS(x,A,b,e)

1. p <= e+1 # by IH
   b <= m+1 <= p # since b<= m (by exercise)
3. p <= e ==> A[p] >= x # by IH
2. b < p # always true, since p >= m+1 > b
   either p = m+1 OR p > m+1
   case p > m+1 ==> A[p-1] < x # by 2. of IH
   case p = m+1 ==> A[p-1] = A[m] < x (by assumption of this case)

what could possibly go wrong?

- $m = \lceil \frac{e+b}{2.0} \rceil$
- $x < A[m]$
- $\ldots$
- Either prove correct, or find a counter-example

Notes