CSC236 fall 2018

recurrences...

Danny Heap

heap@cs.toronto.edu   /   BA4270 (behind elevators)
http://www.teach.cs.toronto.edu/~heap/236/F18/
416-978-5899

Using Introduction to the Theory of Computation, Chapter 3

Outline

induction on recurrences

Notes

recursively defined function

define:
$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Write out a few values of $f(n)$. Conjectures?

$P(n) : f(n) < 2^{n+2}$

Recursive definition

This sequence comes up in applied rabbit breeding, the height of AVL trees, and the complexity of Euclid's algorithm for the GCD, and an astonishing number of other places :

$$F(n) = \begin{cases} n & n < 2 \\ F(n-2) + F(n-1) & n \geq 2 \end{cases}$$

What is the sum from $F(0)$ to $F(n)$?

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

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Fibonacci patterns...

what are Fibonacci numbers multiples of?

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Closed form for $F(n)$?

The course notes present a proof by induction that

$$F(n) = \frac{\phi^n - \left(\hat{\phi}\right)^n}{\sqrt{5}}, \quad \phi = \frac{1 + \sqrt{5}}{2}, \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

You can, and should, be able to work through the proof. The question remains, **how** did somebody ever think of ϕ and $\hat{\phi}$?
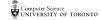
Closed form

Start with the idea that $F(n)$ seems to increase by something **close to** a fixed ratio. Let's try calling that r, and r has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 = r + 1$$

There are two solutions to the quadratic equation: $\phi = r_1$ and $\hat{\phi} = r_2$, but what about the $1/\sqrt{5}$ factor?

If r_1 and r_2 satisfy the recursive definition of $F(n)$, so do linear combinations, and linear combinations give us more freedom:

$$\alpha r_1^n + \beta r_2^n = \alpha r_1^{n-1} + \beta r_2^{n-1} + \alpha r_1^{n-2} + \beta r_2^{n-2}$$

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rabbits get hats

Match up α and β to solutions:

$$
\begin{aligned}
\alpha r_1^0 + \beta r_2^0 &= 0 \quad \Rightarrow \alpha = -\beta \\
\alpha r_1^1 + \beta r_2^1 &= 1 \quad \Rightarrow \alpha(r_1 - r_2) = 1
\end{aligned}
$$

more rabbits...

What about a closed form for

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Notes