term test #1 will be during your Friday tutorial time... rooms to be announced... 3 practice tests under week 5 on the course web site...

CSC236 fall 2018

recurrences...

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Using Introduction to the Theory of Computation, Chapter 3

Outline

induction on recurrences

Notes

recursively defined function

define:

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Write out a few values of $f(n)$. Conjectures?

n:	0	1	2	3	4	5	6	7	8
f(n):	2	7	11	25	47	97	191	385	7???

conjecture: f(n) < 2^{n+2}

$P(n) : f(n) < 2^{n+2}$

Prove by complete induction that \forall n \in \N, P(n).

Let n be an arbitrary natural number. Assume P(0) and ... and P(n-1). I will show that P(n) follows, that is f(n) < 2^{n+2}.

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case n>=2: f(n) = 2f(n-2) + f(n-1) # by definition, since n>= 2
                < 2x2^n + 2^{n+1} # by P(n-2) and P(n-1), since n-1>n-2>=0
                 = 2x2^{n+1} = 2^{n+2}
so P(n) follows in this case.

base case n < 2: f(0) = 2 < 4 = 2^{0+2}, and  f(1) = 7 < 8 = 2^{1+2}, so
                P(n) holds in these cases.
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Recursive definition

Fibonacci sequence

This sequence comes up in applied rabbit breeding, the height of AVL trees, and the complexity of Euclid's algorithm for the GCD, and an astonishing number of other places :

$$F(n) = \begin{cases} n & n < 2 \\ F(n-2) + F(n-1) & n \geq 2 \end{cases}$$

What is the sum from $F(0)$ to $F(n)$?

n:	0	1	2	3	4	5	6	7	8	9	10
F(n):	0	1	1	2	3	5	8	13	21	34	55
sum:	0	1	2	4	7	12	20	33	54		

conjecture: F(0) + ... + F(n) = F(n+2) - 1

Fibonacci numbers

P(n): F(0) + ... + F(n) = F(n+2) - 1

What is $\sum_{i=0}^{i=n} F(i)$?

Prove by simple induction that \forall n \in \N, P(n).

base case, n = 0: Then F(0) + ... + F(0) = F(0) = 0 = 1 - 1 = F(0+2) - 1. So P(0) holds.

Let n be an arbitrary natural number. Assume P(n). I will show that P(n+1) follows.

F(0) + ... + F(n) + F(n+1) = [F(0) + ... + F(n)] + F(n+1)
= F(n+2) - 1 + F(n+1) # by P(n)
= F(n+3) -1 = F(n+1+2) - 1 # by definition, since n>=0 => n+3>=2

So P(n+1) follows in this case.

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

Fibonacci patterns...

P(n): F(3n) is even

what are Fibonacci numbers multiples of?

n:	0	1	2	3	4	5	6	7	8	9	10
F(n):	0	1	1	2	3	5	8	13	21	34	55

Proof by simple induction:

base case: F(3x0) = F(0) = 0 = 2x0, so P(0) holds.

inductive step: Let n be an arbitrary natural number. Assume P(n), that is F(3n) is even. We want to show that P(n+1) follows, that is F(3(n+1)) is even. Let j \in \Z be such that F(3n) = 2j. Let k = j+F(3n+1). We will show that F(3n + 3) = 2k.

So F(3n + 3) = F(3n+1) + F(3n+2) # by definition, since 3n+3>=2
= F(3n+1) + F(3n) + F(3n+1) # by definition, since 3n+2>=2
= 2F(3n+1) + 2j # by P(n) and choice of j
= 2k
So P(n+1) follows

Fibonacci patterns...

what are Fibonacci numbers multiples of?

Prove that \forall n \in \N, P(n)

Closed form for $F(n)$?

No rabbit, no hat

The course notes present a proof by induction that

$$F(n) = \frac{\phi^n - \left(\hat{\phi}\right)^n}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2}, \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

You can, and should, be able to work through the proof. The question remains, **how** did somebody ever think of ϕ and $\hat{\phi}$?

Closed form

... without rabbit

Start with the idea that $F(n)$ seems to increase by something **close to** a fixed ratio. Let's try calling that r, and r has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 = r + 1$$

There are two solutions to the quadratic equation: $\phi = r_1$ and $\hat{\phi} = r_2$, but what about the $1/\sqrt{5}$ factor?

If r_1 and r_2 satisfy the recursive definition of $F(n)$, so do linear combinations, and linear combinations give us more freedom:

$$\alpha r_1^n + \beta r_2^n = \alpha r_1^{n-1} + \beta r_2^{n-1} + \alpha r_1^{n-2} + \beta r_2^{n-2}$$

r_1^n = r_1^{n-1} + r_1^{n-2} ==> ar_1^n = ar_1^{n-1} + ar_1^{n-2} # matches recursive def
r_2^n = r_2^{n-1} + r_2^{n-2} ==> br_2^n = br_2^{n-1} + br_2^{n-2} # matches recursive def

rabbits get hats

Match up α and β to solutions:

$$\begin{aligned} \alpha r_1^0 + \beta r_2^0 &= 0 && \Rightarrow \alpha = -\beta \\ \alpha r_1^1 + \beta r_2^1 &= 1 && \Rightarrow \alpha(r_1 - r_2) = 1 \end{aligned}$$

==> \alpha = 1/(r_1 - r_2)
\beta = -1/(r_1 - r_2)

more rabbits...

What about a closed form for

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

maybe f(n) is something like a linear combination of some geometric series... any such geometric would have to obey r^n = r^{n-1} + 2r^{n-2} => r^2 = r + 2 ==> r^2 - r - 2 = 0
r_1 = 2   r_2 = -1

a(2)^0 + b(-1)^0 = 2 ==> a + b = 2 ==> a = 2 - b
a(2)^1 + b(-1)^1 = 7 ==> 2(2-b) - b = 7 => 4 - 3b = 7 => b = -1, a = 3

f(n) = 3(2)^n - (-1)^n

Notes