a1 due this Friday at 3... office hours MW 2:30--4:30, F 1:00--3:30...

Extra office hours: Thursday noon--5 in BA2230 (various course TAs)
Friday 1--3 (me and a course TA)

CSC236 fall 2018

structural induction, well ordering

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Using Introduction to the Theory of Computation,
Section 1.2–1.3

Computer Science
UNIVERSITY OF TORONTO

Outline

Structural induction

Well-ordering

Define sets inductively

. . . so as to use induction on them later!

E.g., one way to define the natural numbers:

$\mathbb{N}$: The **smallest** set such that

1. $0 \in \mathbb{N}$
2. $n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}$.

By **smallest** I mean $\mathbb{N}$ has no proper subsets that satisfy these two conditions. If I leave out **smallest**, what other sets satisfy the definition? Z, R, C, Q

What can you do with it?

The definition on the previous page defined the simplest natural number (0) and the rule to produce new natural numbers from old (add 1). Proof using Mathematical Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is

1. show that $P(0)$ is true for basis, 0 — basis
2. Prove that $\forall n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$. — inductive step

Other structurally-defined sets

Define $\mathcal{E}$: The **smallest** set such that

- $x, y, z \in \mathcal{E}$
- $e_1, e_2 \in \mathcal{E} \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2)$, and $(e_1 \div e_2) \in \mathcal{E}$.

Form some expressions in $\mathcal{E}$. Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators (symbols from $\{+, \times, \div, -\}$). Make a conjecture.

x \in E	1 variable, 0
(x+x) \in E	2 variables, 1 operator
((x+x)+x) \in E	3 variables, 2 operators
(((x+x)+x)-y)	4 variables, 3 operators

conjecture? vr(e) = op(e) +1

Structural induction

$P(e)$: $\mathrm{vr}(e) = \mathrm{op}(e) + 1$

To prove that a property is true for all $e \in \mathcal{E}$, parallel the recursive set definition:

verify base case(s): Show that the property is true for the simplest members, $\{x, y, z\}$, that is show $P(x)$, $P(y)$, and $P(z)$.

inductive step: Let e_1 and e_2 be arbitrary elements of $\mathcal{E}$.

Assume $H(\{e_1, e_2\}) : P(e_1)$ and $P(e_2)$, that is e_1 and e_2 have the property.

show that $C(\{e_1, e_2\})$ follows:

All possible combinations of e_1 and e_2 have the property, that is $P((e_1 + e_2))$, $P((e_1 - e_2))$, $P((e_1 \times e_2))$, and $P((e_1 \div e_2))$.

Structural induction

$P(e)$: $\mathrm{vr}(e) = \mathrm{op}(e) + 1$

Prove $\forall e \in \mathcal{E}, P(e)$

Proof by structural induction.

basis: Let a \in {x, y, z}. Then a has one variable (itself) and 0 operators, so vr(a) = 1 = 0 + 1 = op(a) + 1. So P(a) holds.

inductive step: Let e_1, e_2 \in E. Let @ \in {+, -, x, /}. Assume P(e_1) and P(e_2). We will show that P((e_1@e_2)) follows, that is vr((e_1@e_2)) = op((e_1@e_2)) + 1.

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vr((e_1@e_2)) = vr(e_1) + vr(e_2)  # haven't changed any variables
              = [op(e_1) + 1 + op(e_2)] + 1   # by P(e_1) and P(e_2)
              = op((e_1@e_2)) + 1             # since (e_1@e_2) has 1 more operator
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So P((e_1@e_2)) follows.

More structural induction

how nested elements are

Define the heights, $h(x) = h(y) = h(z) = 0$, and $\mathrm{h}((e_1 \odot e_2))$ as $1 + \max(\mathrm{h}(e_1), \mathrm{h}(e_2))$, if $e_1, e_2 \in \mathcal{E}$ and $\odot \in \{+, \times, \div, -\}$.

What's the connection between the number of variables and the height?

x 1 variable, height 0

(((x*y)-(z*x))+((y*z)-(x*y))) 2 variables, height 1

4 variable, height 2

8 variables, height 3

conjecture: vr(e) <= 2^{h(e)}

More structural induction

$P(e) : \mathrm{vr}(e) \leq 2^{\mathrm{h}(e)}$

Proof by structural induction.

basis: Let a \in {x, y, z}. Then a has one variable, and height of 0. So vr(a) = 1 <= 1 = 2^{h(a)}, and P(a) holds.

inductive step: Let e_1, e_2 \in E. Assume P(e_1) and P(e_2). Let @ \in {+, -, *, //}. We will show that P((e_1@e_2)), that is vr((e_1@e_2)) <= 2^{h((e_1@e_2))}.

vr((e_1@e_2)) = vr(e_1) + vr(e_2) # didn't change variables
<= 2^{h(e_1)} + 2^{h(e_2)} # by P(e_1) and P(e_2)
<= 2 * 2^{max(h(e_1), h(e_2))} = 2^{max(h(e_1), h(e_2)) + 1}
= 2^{h((e_1@e_2))}
So P((e_1@e_2)) follows.

Well-ordering example

$\forall n, m \in \mathbb{N}, n \neq 0,\ R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$ has a smallest element

NB: notice that we construct set R of natural numbers that is connected to pairs (q, r)

15//6 = 2 15%6=3 15\div 6 = 2R3

This is the main part of proving the existence of a unique quotient and remainder:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

The course notes use Mathematical Induction. Well-ordering is shorter and clearer.

Principle of well-ordering

Every non-empty subset of $\mathbb{N}$ has a smallest element

open interval (0, 1) \subseteq R

Is there something similar for $\mathbb{Q}$ or $\mathbb{R}$? {1/n : n \in \N^+}

For a given pair of natural numbers $m, n \neq 0$ does the set R satisfy the conditions for well-ordering?

$$R = \{r \in \mathbb{N} \mid \exists q \in \mathbb{N}, m = qn + r\}$$

If so, we still need to be sure that the smallest element, r' has

1. $0 \leq r' < n$
2. That q' and r' are unique — no other natural numbers would work

won't prove part 2. --- it requires neither well-ordering nor induction

. . . in order to have

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists! q', r' \in \mathbb{N}, m = q'n + r' \wedge 0 \leq r' < n$$

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N} - \{0\}, \exists q, r \in \mathbb{N}, m = qn + r \wedge 0 \leq r < n$$

Use: every non-empty subset of $\mathbb{N}$ has a smallest element

Let m \in \N. Let n \n \N-{0}. Let R = {r \in \N : \exists q \in \N, m = qn + r}. Then m \in R, with corresponding q = 0, so R is non-empty. Then, by PWO, R has a smallest element. Let r' be the smallest element of R, with corresponding q', so m = q'n + r'.

Then r' < n. In order to see this, suppose (for the sake of contradiction) that r' >= n.

Then r'-n >= 0, so m = q'n + r' = (q'+1)n + (r'-n) and r'-n >= 0.
Then r'-n \in R, but ooops r'-n < r' ----------><---------- contradiction!!!!! r' is smallest element of R

Assuming r' >= n leads to a contradiction, so that assumption is false.

$P(n)$: Every round-robin tournament with n players with a cycle has a 3-cycle

T>D>C>I>G>T

T>T ???!!?

T>D>T ????

Use: every non-empty subset of $\mathbb{N}$ has a smallest element

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

If there is a cycle $p_1 > p_2 > p_3 \ldots > p_n > p_1$, can you find a shorter one?

Every non-empty subset of $\mathbb{N}$ has a smallest element

$P(n)$: Every round-robin tournament with n players that has a cycle has a 3-cycle

Claim: $\forall n \in \mathbb{N} - \{0, 1, 2\}, P(n)$.

Let T be a tournament with n>=3 contestants. Assume T has at least one cycle.
Let C = {c \in \N-{0, 1, 2} : c is the length of a cycle in T}. C is not empty, since we have assume that T has at least one cycle. By PWO C has a least element. Let c' be the smallest element of C.

For sake of contradiction assume that c' is not equal to 3. Since there are no 0-, 1-, or 2- cycles, this means assuming that c'>3.

Then we have a cycle p1>p2>p3>...p_{c'} > p1.
There are 2 cases to consider: either p1>p3 XOR p3>p1
case p1>p3: Then there is a cycle p1>p3>....p_{c'}>p1, of length c'-1 ----><---- contradiction, c' is the shortest length
case p3>p1: Then there is a cycle p1>p2>p3>p1, of length 3 ----><----- 3 < c'

In both possible cases there is a contradiction.
Assuming c'>3 leads to a contradiction, so that assumption is false.

Notes

Define RxR as the smallest set such that

1. (0, 1) and (1, 0) \in RxR
2. if v_1, v_2 \in RxR and x_1, x_2 \in R, then
   x_1*v_1 + x_2*v_2 \in RxR