open season: when I last checked there were a few enrollment spots open
--- second tutorial exercise now posted
--- office hours today, Wednesday 2:30--4:30 in BA2230, Friday 1:30--3:30, same place

CSC236 fall 2018

complete induction

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use Introduction to the Theory of Computation,
Section 1.3
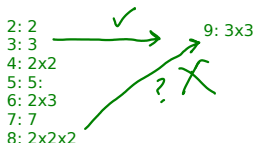
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Complete Induction

another flavour needed

Every natural number greater than 1 has a prime factorization

unique prime factorization theorem *follows* from this

Try some examples

2: 2
3: 3
4: 2x2
5: 5:
6: 2x3
7: 7
8: 2x2x2

9: 3x3

?

How does the factorization of 8 help with the factorization of 9?

notational convenience...

I will use (though you don't have to) the following:
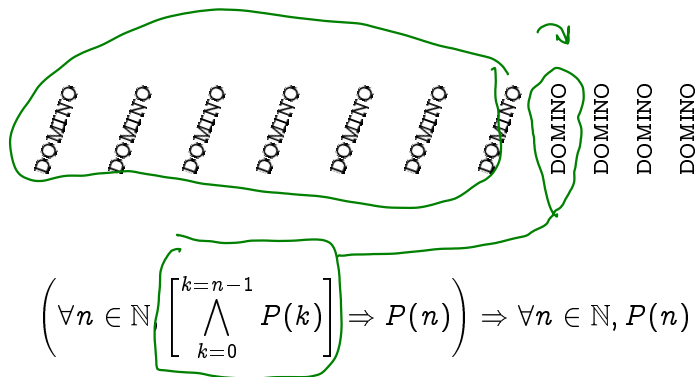
$$\bigwedge_{k=0}^{k=n-1} P(k)$$

... as equivalent to

$$\forall k \in \mathbb{N}, k < n \Rightarrow P(k)$$

More dominoes

$$\left(\forall n \in \mathbb{N} \left[\bigwedge_{k=0}^{k=n-1} P(k) \right] \Rightarrow P(n) \right) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always imply the current case
then all cases are true   what happens when n = 0?

complete induction outline

inductive step: state inductive hypothesis $H(n)$

derive conclusion $C(n)$: show that $C(n)$ follows from $H(n)$, indicating where you use $H(n)$ and why that is valid

verify base case(s): verify that the claim is true for any cases not covered in the inductive step

Wait! isn't that the same outline as simple induction?

Yes, we just modify the inductive hypothesis, $H(n)$ so that it assumes the main claim for every natural number from the starting point up to $n - 1$, and the conclusion, $C(n)$, is now the main claim for n.

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watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ \left[f(\lfloor\sqrt{n}\rfloor)\right]^2 + 2f(\lfloor\sqrt{n}\rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture

f(0) = 1   f(6, ..., 15) = 15
f(1) = 1   f(16) = 255
f(2) = 3   conjecture time!
f(3) = 3
f(4) = 15
f(5) = 15

For all natural numbers $n > 1$, $f(n)$ is a multiple of 3?

use the `complete induction outline`

P(n): f(n) is a multiple of 3

Prove for all natural numbers n > 1, P(n)

inductive step: Let n be a natural number greater than 1. Assume P(2) and ... P(n-1). I will show that P(n) follows, that is f(n) is a multiple of 3.

There are two cases to consider: n < 4 and n >= 4.

case n >= 4. Notice that floor(sqrt(n)) is greater than 1 and less than n, since n >= 4 means sqrt(n) >= 2, also n > 1, so n^2 > n, so n > sqrt(n). So I may use the assumption P(floor(sqrt(n))), in other words f(floor(sqrt(n))) is a mutiple of 3. Let k be a natural number such that f(floor(sqrt(n))) = 3k. So, f(n) = (3k)^2 + 2(3k), by definition, since n > 1. Thus f(n) = 3(3k^2 + 2k), so P(n) follows in this case.

base case n \in {2, 3}: So f(n) = 3 = 3x1, so P(n) follows in this case.

So P(n) follows in both possible cases.

zero pair free binary strings, zpfbs...

Denote by $zpfbs(n)$ the number of binary strings of length n that contain no pairs of adjacent zeros. What is $zpfbs(n)$ for the first few natural numbers n?

First, an extra step, using a function f(n):
  f(0) = 1
  f(1) = 2
  f(2) = 3
  f(3) = 5
  f(4) = 8
  f(5) = 13
  f(6) = 21

P(n): The number of zero-pair-free binary strings of length n is f(n).

$f(n) = f(n-1) + f(n-2)$ if n >=2
$f(0) = 1, f(1) = 2$

b0 b1 b2 .... bn-1 1 ... f(n-1)
b0 b1 b2 ..... bn-2 1 0   f(n-2)

what is $zpfbs(n)$?

use the complete induction outline

P(n): The number of zero-pair-free binary strings of length n is f(n).

Prove, by complete induction that for all natural numbers, P(n).

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Let n be a natural number. Assume P(0) ... P(n-1). I will show that P(n) follows, that is the number of zero-pair-free binary strings of length n is f(n).

There are 3 cases to consider.

base case n = 0: There is exactly one binary string of length 0, that is "", and it has no pairs of zeros. So P(0) follows.

base case n = 1: There are exactly 2 binary strings of length 1: "1" and "0", and neither of them have adjacent zeros. So P(1) follows.

case n>= 2: Partition the zero-pair-free binary strings of length n into those that end with a "1" and those that end with a "0". Those that end with 1 are simply those of length n-1 with a 1 appended, which changes nothing. By P(n-1) there are f(n-1) of these, since 0 <= n-1 < n. Notice that those that end in "0" must actually end in "10" in order to be zero-pair free. By P(n-2) [since n >= 2] we know there are f(n-2) of these. In total there are f(n-1) + f(n-2) = f(n) zero-pair-free binary strings of length n, so P(n) follows in this case.

P(n) follows in each of the possible cases.

Every natural number greater than 1 has a prime factorization

use the complete induction outline

P(n): n can be expressed as a product of primes.

Prove by complete induction:
For all natural numbers greater than or equal to 2, P(n).

prime factorization: representation as product of 1 or more primes

Let n be a natural number greater than 1. Assume P(2) ... P(n-1). I will show that P(n) follows, that is n can be expressed as a product of primes.

There are a couple of cases to consider: n is composite or n is not composite.

case n is composite: By definition, n has a natural number factor f1 with 1 < f1 < n. Let f2 = n/f1, an integer, since f1 is a factor. Since f1 < n, it follows that 1 = f1/f1 < n/f1 = f2. Also since f1 > 1, it follows f2xf1 = n > f2. Thus, by P(f1) and P(f2) they each can be represented as products of primes, and thus n is simply the product of these products. So P(n) follows.

base case n is not composite: By definition n is prime, and is thus its own prime factorization. So P(n) follows

So P(n) follows in both possible cases.

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After a certain natural number n, every postage can be made up by combining 3— and 5— cent stamps

what is the "certain natural number"?

After a certain natural number n, every postage can be made up by combining 3— and 5— cent stamps

use the complete induction outline

After a certain natural number n, every postage can be made up by combining 3— and 5— cent stamps

use the complete induction outline

notes...