CSC236 fall 2018

theory of computation

second-best method:
csc236-2018-09@cs.toronto.edu
or (for course content)

Danny Heap

best method:
face-to-face after lecture, at
office hour, in office

heap@cs.toronto.edu / BA4270 (behind elevators)

http://www.teach.cs.toronto.edu/~heap/236/F18/

416-978-5899

not the best method

use Introduction to the Theory of Computation, Section 1.2

Outline

why reason about computing?

- you're not just hackers anymore...

  sometimes you need to analyze code *before* it run... sometimes it will never be run

- can you test **everything**?

  there are infinitely many integer, string, list inputs...

- careful, you might get to like it... (?!*)

  it's happened before...

how to reason about computing

- it's messy...

  you need to draft, re-draft, ..., many drafts
  you need to follow, and often abandon, blind alleys

- it's art...

  strive for correctness, clarity, surprise, humour, pathos...

how to do well

- read the course information sheet as a two-way promise
- question, answer, record, synthesize

  you could annotate the blank, aka vanilla, slides

- collaborate with respect

  choose respectful collaborators who will challenge your ideas

we behave as though you already know...

- Chapter 0 material from *Introduction to Theory of Computation*
- CSC165 material, especially proofs and big-Oh material
- But you can *relax* the structure a little (more on this later)
- recursion, efficiency material from CSC148

by December you'll know...

- understand, and use, several flavours of induction

  impress your friends...

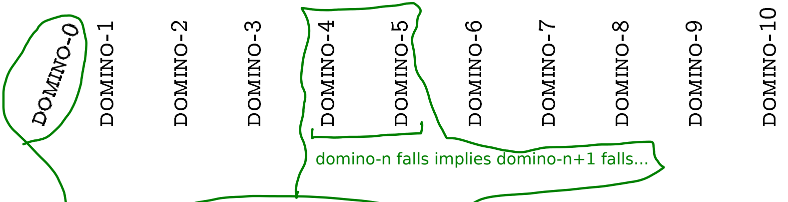
- complexity and correctness of programs — both recursive and iterative

  proofs by induction...

- formal languages, regular languages, regular expressions

  first taste of formal language theory...

domino fates foretold



$$[P(0) \wedge (\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1))] \implies \forall n \in \mathbb{N}, P(n)$$

If the initial case works,
and each case that works implies its successor works,
then all cases work

simple induction outline

inductive step: introduce n and inductive hypothesis $H(n)$

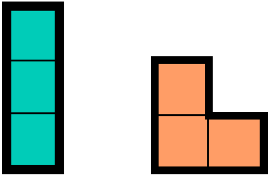
derive conclusion $C(n)$: show that $C(n)$ follows from $H(n)$, indicating **where** you use $H(n)$ and why that is valid

verify base case(s): verify that the claim is true for any cases not covered in the inductive step

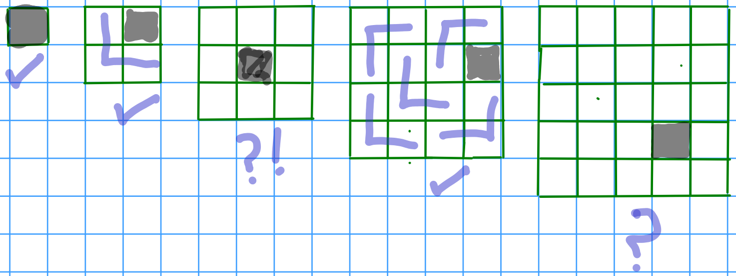
in simple induction $C(n)$ is just $H(n+1)$

trominoes

see https://en.wikipedia.org/wiki/Tromino



Can an $n \times n$ square grid, **with one subsquare removed**, be tiled (covered without overlapping) by "chair" trominoes?



trominoes

$P(n)$ a $2^n \times 2^n$ square grid, **with one subsquare removed**, can be tiled (covered without overlapping) by "chair" trominoes?

base case: let's argue about where to start...

A 2^0 x 2^0 square grid with one arbitrary subsquare removed is simply... empty space! So it can be tiled with... zero chairs. This verifies P(0).

trominoes

$P(n)$ a $2^n \times 2^n$ square grid, **with one subsquare removed**, can be tiled (covered without overlapping) by "chair" trominoes?

inductive step: Let n be an aribitary, fixed, natural number. Assume P(n), that is a 2^n x 2^n square grid with one arbitrary subsquare removed can be tiled with chairs. I will prove P(n+1), that is a 2^{n+1} x 2^{n+1} square grid with one arbitrary subsquare removed can be tiled with chairs.
Let G be a 2^{n+1} x 2^{n+1} square grid with one subsquare removed. Notice that G can be decomposed into 4 2^n x 2^n disjoint quadrant grids. We may assume, without loss of
generality (WLOG), that the removed subsquare is in the upper-right quadrant, since we can always rotate G to make it so, and then rotate G back after we've tiled it.
By P(n) we can tile the upper-right quadrant, minus the missing square, with chairs.f
By P(n) three more times, we can tile the remaining quadrants, omitting for a moment the subsquares of each that are adjacent to the centre of G, with chairs.
The briefly omitted three subsquare form a chair, so we can complete the tiling of G (minus that original missing square) by chairs. So P(n+1) follows.

$3^n \geq n^3$?

scratch work: check for a few values of n

3^0 = 1 >= 0 = 0^3...okay!
3^1 = 3 >= 1 = 1^3...okay!
3^2 = 9 >= 8 >= 2^3...okay!
3^3 = 27 >= 27 = 3^3...okay!!
3^4 = 81 >= 64 = 4^3...okay!
3^{-1} = 1/3 >= -1 = -1^3...okay!!!
3^2.5   <   2.5^3...not okay!!

$3^n \geq n^3$

define P(n): 3^n >= n^3

use the simple induction outline

scope of n restricted here...

Let n be an arbitrary, fixed, natural number that is no smaller than 3.
Assume P(n), that is 3^n >= n^3. I will prove that P(n+1)
follows, that is 3^{n+1} >= (n+1)^3.

3^{n+1} = 3 x 3^n >= 3 x n^3, by P(n)
= n^3 + n^3 + n^3
>= n^3 + 3n^2 + n^3 # since n >= 3
>= n^3 + 3n^2 + 3n + 6n # since n>= 3
>= n^3 + 3n^2 + 3n + 1 # n>= 3 => 6n >= 1
= (n+1)^3 # by binomial theorem
That is, P(n+1) follows.

base cases are those natural numbers that cannot be reached by inductive step, so P(0), P(1), P(2), and P(3). Of these, only P(3) helps in inductive step!

For every $n \in \mathbb{N}$, $12^n - 1$ is a multiple of 11

scratch work: substitute a few values for n

For every $n \in \mathbb{N}$, $12^n - 1$ is a multiple of 11

use the simple induction outline

For every $n \in \mathbb{N}$, $12^n - 1$ is a multiple of 11

use the simple induction outline

The units digit of 7^n is one of 1, 3, 7, or 9

scratch work: substitute a few values for n

The units digit of 7^n is one of 1, 3, 7, or 9

use the simple induction outline

The units digit of 7^n is one of 1, 3, 7, or 9

use the simple induction outline

What about: the units digit of 7^n is one of 1, 2, 3, 7, or 9

use the simple induction outline

is the claim still true? What happens if you add this other case to the inductive step?

how **not** to do simple induction

does a graph $G = (V, E)$ with $|V| > 0$ have $|E| = |V| - 1$?

base case: easy

inductive step: what happens if you try to extend an arbitrary graph with $|V| = n$ to one with $|V'| = n + 1$?

variations: what happens if you restrict the claim to connected graphs? what about acyclic connected graphs?

take-home: decompose rather than construct... except in structural induction (later)

Notes