CSC236 Fall 2016
Assignment #1: induction
due October 7th, 10 p.m.

The aim of this assignment is to give you some practice with various forms of induction. For each question below you will present a proof by induction. For full marks you will need to make it clear to the reader that the base case(s) is/are verified, that the inductive step follows for each element of the domain (typically the natural numbers), where the inductive hypothesis is used and that it is used in a valid case.

Your assignment must be typed to produce a PDF document (hand-written submissions are not acceptable). You may work on the assignment in groups of 1, 2, or 3, and submit a single assignment for the entire group on MarkUs

1. Consider the Fibonacci-esque function g:

$$g(n) = \begin{cases} 1, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ g(n-2) + g(n-1) & \text{if } n > 1 \end{cases}$$

   Use complete induction to prove that if n is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$. You may **not** derive or use a closed-form for $g(n)$ in your proof.

2. Suppose B is a set of binary strings of length n, where n is positive (greater than 0), and no two strings in B differ in fewer than 2 positions. Use simple induction to prove that B has no more than 2^{n-1} elements.

3. Define T as the smallest set of strings such that:

   (a) "b" $\in T$

   (b) If $t_1, t_2 \in T$, then $t_1 +$ "ene" $+ t_2 \in T$, where the $+$ operator is string concatenation.

   Use structural induction to prove that if $t \in T$ has n "b" characters, then t has $2n - 2$ "e" characters.

4. On page 79 of the Course Notes the quantity $\phi = (1 + \sqrt{5})/2$ is shown to be closely related to the Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that ϕ is irrational.

   (a) Show that $\phi(\phi - 1) = 1$.

   (b) Rewrite the equation in the previous step so that you have ϕ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, $+$ or $-$, and ϕ. There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.

(c) Assume, for a moment, that there are natural numbers m and n such that $\phi = n/m$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, $+$ or $-$, m and n.

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains m. Use the Principle of Well-Ordering, plus one of the two expressions for ϕ from the previous step to derive a contradiction.

(e) Combine your assumption and contradiction from the previous step into a proof that ϕ cannot be the ratio of two natural numbers. Extend this to a proof that ϕ is irrational.

5. Consider the function f, where $3 \div 2 = 1$ (integer division, like 3//2 in Python):

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number n greater than 2, $f(n)$ is a multiple of 7. **NB**: Think carefully about which natural numbers you are justified in using the inductive hypothesis for.