

CSC236 tutorial exercises, Week #7

Sample Solutions

1. Consider the recurrence relation

$$T(n) = \begin{cases} 1 & n = 1 \\ 1 + T\left(\left\lceil \frac{n}{2} \right\rceil\right) & n > 1 \end{cases}$$

Prove that $T(n)$ is non-decreasing.

Sample solution.

Let $P(n)$ denote $T(n) \leq T(n + 1)$.

Proof by strong induction.

Basis step.

$P(1)$ holds since $T(1) = 1 \leq T(2) = 1 + T\left(\left\lceil \frac{2}{2} \right\rceil\right) = 1 + 1 = 2$.

Inductive step.

Assume $P(i)$ holds where $1 \leq i < k$ for an arbitrary $k > 1 \in \mathbb{N}$; i.e., $T(i) \leq T(i + 1)$.

We must show $T(k) \leq T(k + 1)$.

There are two cases: either k is odd or k is even.

Case 1. When k is odd, $\left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil$ by definition of ceiling.

$$T(k) = 1 + T\left(\left\lceil \frac{k}{2} \right\rceil\right) \quad \text{by definition of } T \text{ as } k > 1$$

$$T(k) = 1 + T\left(\left\lceil \frac{k+1}{2} \right\rceil\right) \quad \text{as } k \text{ is odd and by IH, since } 1 \leq \left\lceil \frac{k}{2} \right\rceil < k$$

$$T(k) = T(k + 1) \quad \text{by definition of } T \text{ as } k + 1 > 1$$

Case 2. When k is even, $\left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{k}{2} \right\rceil + 1$ by definition of ceiling

$$T(k + 1) = 1 + T\left(\left\lceil \frac{k+1}{2} \right\rceil\right) \quad \text{by definition of } T \text{ as } k > 1$$

$$T(k + 1) = 1 + T\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \quad \text{as } k \text{ is even}$$

$$T(k + 1) \geq 1 + T\left(\left\lceil \frac{k}{2} \right\rceil\right) \quad \text{by IH since } 1 \leq \left\lceil \frac{k}{2} \right\rceil < k \quad \forall k > 1 \in \mathbb{N}$$

$$T(k + 1) \geq T(k) \quad \text{by definition of } T \text{ as } k > 1$$

□

2. Use repeated substitution (unwinding) to find a closed form for the recurrence S when n is a power of 3.

$$S(n) = \begin{cases} 1 & n < 3 \\ n^2 + a_1 S\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + a_2 S\left(\left\lfloor \frac{n}{3} \right\rfloor\right) & n > 2 \end{cases}$$

where $a_1, a_2 \geq 0 \in \mathbb{N}$ and $a_1 + a_2 = 3$.

Sample Solution.

Assume $\hat{n} = 3^k$ where $k \in \mathbb{N}$. Since $\left\lfloor \frac{\hat{n}}{3} \right\rfloor = \left\lfloor \frac{\hat{n}}{3} \right\rfloor = \frac{\hat{n}}{3}$, and $a_1 + a_2 = 3$

$$S(\hat{n}) = \begin{cases} 1 & \hat{n} = 1 \\ \hat{n}^2 + 3S\left(\frac{\hat{n}}{3}\right) & \hat{n} > 1 \end{cases}$$

$$S(\hat{n}) = \hat{n}^2 + 3S\left(\frac{\hat{n}}{3}\right)$$

$$= \hat{n}^2 + 3\left(\left(\frac{\hat{n}}{3}\right)^2 + 3S\left(\frac{\hat{n}}{3^2}\right)\right) \quad \text{by plugging in } \frac{\hat{n}}{3} \text{ to } S(\hat{n})$$

$$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2 S\left(\frac{\hat{n}}{3^2}\right)$$

$$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2 \left(\left(\frac{\hat{n}}{3^2}\right)^2 + 3S\left(\frac{\hat{n}}{3^3}\right)\right) \quad \text{by plugging in } \frac{\hat{n}}{3^2} \text{ to } S(\hat{n})$$

$$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2 \left(\frac{\hat{n}}{3^2}\right)^2 + 3^3 S\left(\frac{\hat{n}}{3^3}\right)$$

...

after k steps

$$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2 \left(\frac{\hat{n}}{3^2}\right)^2 + 3^3 \left(\frac{\hat{n}}{3^3}\right)^2 + \dots + 3^{k-1} \left(\frac{\hat{n}}{3^{k-1}}\right)^2 + 3^k S\left(\frac{\hat{n}}{3^k}\right)$$

$$= \hat{n}^2 + 3\left(\frac{\hat{n}}{3}\right)^2 + 3^2 \left(\frac{\hat{n}}{3^2}\right)^2 + 3^3 \left(\frac{\hat{n}}{3^3}\right)^2 + \dots + 3^{k-1} \left(\frac{\hat{n}}{3^{k-1}}\right)^2 + \hat{n} \quad \text{since } \hat{n} = 3^k, \text{ and } S(1) = 1$$

$$= \hat{n}^2 \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{k-1}}\right) + \hat{n} \quad \text{factoring out } \hat{n}^2$$

$$= \hat{n}^2 \frac{1 - \left(\frac{1}{3}\right)^k}{1 - \frac{1}{3}} + \hat{n} \quad \text{from geometric series}$$

$$= \hat{n}^2 \left(\frac{1}{2/3} - \frac{1/3^k}{2/3}\right) + \hat{n} = \frac{3}{2} \hat{n}^2 \left(1 - \frac{1}{\hat{n}}\right) + \hat{n} = \frac{3}{2} \hat{n}(\hat{n} - 1) + \hat{n}$$

Now we should prove $S_r(\hat{n}) = S_c(\hat{n})$ where

$$S_r(\hat{n}) = \begin{cases} 1 & n < 3 \\ \hat{n}^2 + a_1 S_r\left(\left\lfloor \frac{\hat{n}}{3} \right\rfloor\right) + a_2 S_r\left(\left\lfloor \frac{\hat{n}}{3} \right\rfloor\right) & n > 2 \end{cases}$$

where $a_1, a_2 \geq 0 \in \mathbb{N}$, $a_1 + a_2 = 3$, and $S_c(\hat{n}) = \frac{3}{2}\hat{n}(\hat{n} - 1) + \hat{n}$ where $\hat{n} = 3^k$.

Proof by strong induction.

Basis step.

$$S_r(1) = 1 = S_c(1) = \frac{3}{2}1(1 - 1) + 1 = 1$$

Inductive step.

Assume $S_r(i) = S_c(i)$ where $1 \leq i < \hat{k}$ for $\hat{k} \geq 3 \in \mathbb{N}$ where \hat{i} and \hat{k} are powers of 3.

We must show $S_r(\hat{k}) = S_c(\hat{k})$

$$S_r(\hat{k}) = \hat{k}^2 + a_1 S_r\left(\left\lceil \frac{\hat{k}}{3} \right\rceil\right) + a_2 S_r\left(\left\lfloor \frac{\hat{k}}{3} \right\rfloor\right) \quad \text{by definition of } S_r \text{ when } \hat{k} > 2$$

$$S_r(\hat{k}) = \hat{k}^2 + (a_1 + a_2) S_r\left(\frac{\hat{k}}{3}\right) \quad \text{since } \hat{k} \text{ is a power of } 3 \quad \left\lceil \frac{\hat{k}}{3} \right\rceil = \left\lfloor \frac{\hat{k}}{3} \right\rfloor = \frac{\hat{k}}{3}$$

$$S_r(\hat{k}) = \hat{k}^2 + 3 S_r\left(\frac{\hat{k}}{3}\right) \quad \text{since } a_1 + a_2 = 3$$

$$S_r(\hat{k}) = \hat{k}^2 + 3 S_c\left(\frac{\hat{k}}{3}\right) \quad \text{by IH, since } 1 \leq \frac{\hat{k}}{3} < \hat{k} \text{ when } \hat{k} \geq 3$$

$$S_r(\hat{k}) = \hat{k}^2 + 3 \cdot \left(\frac{3}{2} \frac{\hat{k}}{3} \left(\frac{\hat{k}}{3} - 1 \right) + \frac{\hat{k}}{3} \right) \quad \text{by definition of } S_c$$

$$S_r(\hat{k}) = \hat{k}^2 + \frac{\hat{k}^2}{2} - \frac{3}{2} \hat{k} + \hat{k} = \frac{3}{2} \hat{k}^2 - \frac{3}{2} \hat{k} + \hat{k} = \frac{3}{2} \hat{k}(\hat{k} - 1) + \hat{k} = S_c(\hat{k})$$

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