CSC236 tutorial exercises, Week #3
sample solutions

Work on small examples until you believe the claim (and find k for the second question). Students should try writing out the inductive hypothesis, without completing the inductive step, and discussing whether it is reasonable.

For the first question, the solution should be in terms of an inequality, rather than proving and using an equality.

1. Full binary trees are binary trees where all internal nodes have 2 children. Prove that any full binary tree with more than 1 node has no more than twice as many leaves as internal nodes. Use complete induction on the total number of nodes.

   Sample solution: Proof by complete induction on the number of nodes in the binary tree.

   inductive step: Let $n \in \mathbb{N}, n > 1$. Assume $H(n)$: $\forall i \in \mathbb{N}, 2 \leq i < n$ every full binary tree with i nodes has no more than twice as many leaves as internal nodes.

   show $C(n)$: Every full binary tree with n nodes has no more than twice as many leaves as internal nodes.

   Case: no full binary trees with n nodes exist: $C(n)$ is vacuously true (e.g. $n = 2$ or other even numbers greater than 0). # For the purposes of this proof, there is no need to prove the absence of all full binary trees with an even number of nodes greater than 0, although the proof is not too hard.

   Case: one or more full binary trees with n nodes exist: Let T be a full binary tree with n nodes. There are three cases to consider:

   Case: both of T's sub-trees are single-node trees: T has one interior node (the root) and two leaves, and $2 \leq 2 \times 1$, so $C(n)$ is true in this case.

   Case: one of T's subtrees is a single-node tree, and the other has more than 1 node: Let the number of internal nodes in T's subtree with more than 1 node be i_c, and the number of leaves be l_c. Since this subtree has fewer than n and more than 1 node, by $H(n)$ we know that $l_c \leq 2i_c$. The number of leaves in T are $l_c + 1$, the leaves of the subtree with more than 1 node and the single-node subtree. The number of internal nodes in T are $i_c + 1$, the internal nodes of the subtree with more than one node plus the root. Summing leaves and internal nodes and comparing them we get:

   $$l_c + 1 \leq 2i_c + 1 \leq 2(i_c + 1)$$

   So $C(n)$ is true in this case

Case: both of T's subtrees have more than 1 node: Let the number of internal nodes and leaves of the left subtree be i_L and l_L, respectively. Let the number of internal nodes and leaves of the right subtree be i_R and l_R, respectively. Since each subtree has fewer than n and more than 1 node, by $H(n)$ we know that $l_L \leq 2i_L$ and $l_R \leq 2i_R$. Summing T's leaves, and comparing them to number of T's internal nodes, $i_l + i_R + 1$ (the root is an internal node):

$$l_L + l_R \leq 2i_L + 2i_R = 2(i_L + i_R) \leq 2(i_L + i_R + 1)$$

So $C(n)$ is true in this case.

In every possible case $C(n)$ follows from $H(n)$.

2. Use Complete Induction to show that postage of exactly n cents can be made using only 6-cent and 7-cent stamps, for every natural number n greater than k (you will have to discover the value of k).

Sample solution: Proof by complete induction that $\forall n \in \mathbb{N}, n > 29$ postage of n cents can be made with 6-cent and 7-cent stamps. (I guessed $k = 29$ by experimenting).

inductive step: Let $n \in \mathbb{N}, n > 29$. Assume $H(n)$: $\forall i \in \mathbb{N}, 30 \leq i < n$, postage of i cents can be made with 6-cent and 7-cent stamps.

show $C(n)$: postage of n cents can be made with 6-cent and 7-cent stamps.

Case $n = 30$ (base case): Postage of 30 cents can be formed with 5 6-cent stamps and 0 7-cent stamps.

Case $n = 31$ (base case): Postage of 31 cents can be formed with 4 6-cent stamps and 1 7-cent stamps.

Case $n = 32$ (base case): Postage of 32 cents can be formed with 3 6-cent stamps and 2 7-cent stamps.

Case $n = 33$ (base case): Postage of 33 cents can be formed with 2 6-cent stamps and 3 7-cent stamps.

Case $n = 34$ (base case): Postage of 34 cents can be formed with 1 6-cent stamp and 4 7-cent stamps.

Case $n = 35$ (base case): Postage of 35 cents can be formed with 0 6-cent stamps and 5 7-cent stamps.

Case $n > 35$: Since $n - 6 > 29$ (subtract 6 from the inequality), we know that $29 < n - 6 < n$ and by $H(n)$ postage of $n - 6$ cents can be formed with 6-cent and 7-cent stamps. Add one 6-cent stamp to make postage for n cents with 6-cent and 7-cent stamps.

In every case $C(n)$ follows from $H(n)$.