

CSC236 tutorial exercises, Week #2

sample solutions

Prove the following claims using Mathematical Induction (AKA Simple Induction).

During tutorial: Students should be asked to verify each claim for a few cases, ideally enough to see a pattern. Take a show of hands to see which problem has the largest number puzzled, then work on constructing the inductive step as a group. Try to at least sketch inductive steps for each problem.

I think it's a good idea to ask how many base cases they think they need for the second problem. I've seen students think they need one for each residue modulo 10, so you may need to convince them that once they directly verify the claim for $n = 0$, each other case can be reached.

Also note that I tend to do the inductive step before the base case. My rationale is that you can only really be sure of what base case(s) are required after you've established the implication in the inductive step. Of course, the other order is also correct.

Also note that I don't necessarily name the main claim. I name the inductive hypothesis $H(n)$ and the conclusion it leads to $C(n)$ (they are each, in turn, the main claim with respect to n and $n + 1$, respectively). This is the template I present to my students, and it is the same template for complete induction (but $H(n)$ and $C(n)$ are modified). Again, of course, other presentations are also correct.

1. $\forall n \in \mathbb{N}$, $7^n - 1$ is a multiple of 6.

Sample solution 1: Proof by simple induction:

inductive step: Let n be an arbitrary natural number. Assume $H(n)$: $7^n - 1$ is a multiple of 6.

show that $C(n)$ follows from $H(n)$: Denote by $C(n)$ the claim: $7^{n+1} - 1$ is a multiple of 6.

Let $k \in \mathbb{Z}$, such that $7^n - 1 = 6k$ # since by $H(n)$ $7^n - 1$ is a multiple of 6

$$7^{n+1} - 1 = 7(7^n - 1) + 6 = 7(6k) + 6 = 6(7k + 1)$$

$$7k + 1 \in \mathbb{Z} \text{ # since } 7, k, 1 \in \mathbb{Z} \text{ and } \mathbb{Z} \text{ closed under } \times, +$$

$7^{n+1} - 1$ is a multiple of 6, that is $C(n)$ follows from $H(n)$.

base case: $7^0 - 1 = 1 - 1 = 0 = 6 \times 0$, so the main claim holds for natural number 0.

Sample solution 2: Proof by simple induction:

Let $P(n)$ denote $7^n - 1 = 6m_n$, where $m_n \in \mathbb{Z}$.

basis step: $P(0)$ holds because $7^0 - 1 = 1 - 1 = 0 = 6 \times 0$.

inductive step: Assume $P(k)$ holds for some arbitrary $k \geq 0 \in \mathbb{N}$, that is $7^k - 1 = 6m_k$, where $m_k \in \mathbb{Z}$.

Using the I.H., we must show that $P(k + 1)$ holds too:

$$7^{k+1} - 1 = 7(7^k - 1) + 6 = 7(6m_k) + 6 = 6(7m_k + 1)$$

$$m_{k+1} = 7m_k + 1 \in \mathbb{Z} \text{ # since } 7, m_k, 1 \in \mathbb{Z} \text{ and } \mathbb{Z} \text{ closed under } \times, +$$

This completes the inductive step.

Hence, $\forall n \in \mathbb{N}$, $7^n - 1$ is a multiple of 6.

2. $\forall n \in \mathbb{N}$, the units digit of 7^n is in $\{1, 3, 7, 9\}$.

Sample solution 1: Proof by simple induction.

inductive step: Let n be an arbitrary natural number. Assume $H(n)$: the units digit of 7^n is in $\{1, 3, 7, 9\}$.

show $C(n)$ follows from $H(n)$: Denote by $C(n)$ the claim: The units digit of $7^{n+1} \in \{1, 3, 7, 9\}$.

By $H(n)$ there are natural numbers i, j such that $7^n = 10i + j$ and $j \in \{1, 3, 7, 9\}$. There are four cases to consider:

Case $j = 1$: $7^{n+1} = 7 \times 7^n = 10 \times 7i + 7$, and the units digit $7 \in \{1, 3, 7, 9\}$.

Case $j = 3$: $7^{n+1} = 7 \times 7^n = 70i + 21 = 10(7i + 2) + 1$ and the units digit $1 \in \{1, 3, 7, 9\}$.

Case $j = 7$: $7^{n+1} = 7 \times 7^n = 70i + 49 = 10(7i + 4) + 9$, and the units digit $9 \in \{1, 3, 7, 9\}$.

Case $j = 9$: $7^{n+1} = 7 \times 7^n = 70i + 63 = 10(7i + 6) + 3$, and the units digit $3 \in \{1, 3, 7, 9\}$.

In every possible case, the units digit of $7^{n+1} \in \{1, 3, 7, 9\}$, so $C(n)$ follows from $H(n)$.

base case: $7^0 = 1 \in \{1, 3, 7, 9\}$, so the main claim is verified for natural number 0.

Sample solution 2: Proof by simple induction. We also use the modular arithmetic notation.

Let $P(n)$ denote $7^n \equiv 1, 3, 7, \text{ or } 9 \pmod{10}$.

basis step: $P(0)$ holds because $7^0 \equiv 1 \pmod{10}$.

inductive step: Assume $P(k)$ holds for some arbitrary $k \geq 0 \in \mathbb{N}$, that is $7^k \equiv 1, 3, 7, \text{ or } 9 \pmod{10}$.

Using the I.H., we must show that $P(k+1)$ holds too: since $7^{k+1} \equiv 7^k \times 7 \pmod{10}$, there are four cases to consider:

Case $7^k \equiv 1 \pmod{10}$: By the I.H., $7^{k+1} \equiv 1 \times 7 \pmod{10} \equiv 7 \pmod{10}$.

Case $7^k \equiv 3 \pmod{10}$: By the I.H., $7^{k+1} \equiv 3 \times 7 \pmod{10} \equiv 1 \pmod{10}$.

Case $7^k \equiv 7 \pmod{10}$: By the I.H., $7^{k+1} \equiv 7 \times 7 \pmod{10} \equiv 9 \pmod{10}$.

Case $7^k \equiv 9 \pmod{10}$: By the I.H., $7^{k+1} \equiv 9 \times 7 \pmod{10} \equiv 3 \pmod{10}$.

This completes the inductive step.

Hence, $\forall n \in \mathbb{N}$, $7^n \equiv 1, 3, 7 \text{ or } 9 \pmod{10}$.

3. $\forall n \in \mathbb{N} - \{0, 1, 2, 3\}, 4^n \geq n^4$

Sample solution 1: Proof by simple induction.

inductive step: Let $n \in \mathbb{N}$. Assume $n \geq 4$ and assume $H(n)$: $4^n \geq n^4$.

show $C(n)$ follows from $H(n)$: Denote by $C(n)$ the claim: $4^{n+1} \geq (n+1)^4$.

$$\begin{aligned}
 4^{n+1} &= 4 \times 4^n \geq 4 \times n^4 \# \text{ since by } H(n), 4^n \geq n^4 \\
 &= n^4 + n^4 + n^4 + n^4 \\
 &\geq n^4 + 4n^3 + 16n^2 + 64n \# \text{ since } n \geq 4 \\
 &\geq n^4 + 4n^3 + 6n^2 + 4n + 60n \\
 &> n^4 + 4n^3 + 6n^2 + 4n + 1 \# \text{ since } n \geq 4 > 1/60 \\
 &= (n+1)^4 \# \text{ binomial theorem}
 \end{aligned}$$

That is, $C(n)$ follows from $H(n)$.

base case: $4^4 = 64 \geq 64 = 4^n$, so the main claim holds for natural number 4.

Sample solution 2: Proof by simple induction.

Let $P(n)$ denote $4^n \geq n^4$.

basis step: $P(4)$ holds because $4^4 \geq 4^4$.

inductive step: Assume $P(k)$ holds for some arbitrary $k \geq 4 \in \mathbb{N}$, that is $4^k \geq k^4$.

Using the I.H., we must show that $P(k+1)$ holds too:

$$\begin{aligned} 4^{k+1} &= 4 \times 4^k \geq 4 \times k^4 \# \text{ since by I.H., } 4^k \geq k^4 \\ &= k^4 + k^4 + k^4 + k^4 \\ &\geq k^4 + 4k^3 + 16k^2 + 64k \# \text{ since } k \geq 4 \\ &\geq k^4 + 4k^3 + 6k^2 + 4k + 60k \\ &> k^4 + 4k^3 + 6k^2 + 4k + 1 \# \text{ since } k \geq 4 > 1/60 \\ &= (k+1)^4 \# \text{ binomial theorem} \end{aligned}$$

This completes the inductive step.

Hence, $\forall n \geq 4 \in \mathbb{N}$, $4^n \geq n^4$.