CSC236 fall 2016

recurrences...

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Using Introduction to the Theory of Computation, Chapter 3

Outline

induction on recurrences

Notes

recursively defined function

define:

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Write out a few values of $f(n)$. Conjectures?

$f(n) < 2^{n+2}$

Recursive definition

Fibonacci sequence

This sequence comes up in applied rabbit breeding, the height of AVL trees, and the complexity of Euclid's algorithm for the GCD:

$$F(n) = \begin{cases} n & n < 2 \\ F(n-2) + F(n-1) & n \geq 2 \end{cases}$$

What is the sum of n Fibonacci numbers?

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

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Fibonacci patterns...

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Closed form for $F(n)$?

No rabbit, no hat

The course notes present a proof by induction that

$$F(n) = \frac{\phi^n - \left(\hat{\phi}\right)^n}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2}, \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

You can, and should, be able to work through the proof. The question remains, how did somebody ever think of ϕ and $\hat{\phi}$?

Closed form

. . . without rabbit

Start with the idea that $F(n)$ seems to increase by something close to a fixed ratio. Let's try calling that r, and r has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \Rightarrow r^2 = r + 1$$

There are two solutions to the quadratic equation: ϕ and $\hat{\phi}$, but what about the $1/\sqrt{5}$ factor?

If ϕ and $\hat{\phi}$ are solutions, so are linear combinations:

$$\alpha\phi^n + \beta\hat{\phi}^n = \alpha\phi^{n-1} + \beta\hat{\phi}^{n-1} + \alpha\phi^{n-2} + \beta\hat{\phi}^{n-2}$$

rabbits get hats

Match up α and β to solutions:

$$\begin{aligned} \alpha\phi^0 + \beta\hat{\phi}^0 &= 0 \quad \Rightarrow \alpha = -\beta \\ \alpha\phi^1 + \beta\hat{\phi}^1 &= 1 \quad \Rightarrow \alpha(\phi - \hat{\phi}) = 1 \end{aligned}$$

more rabbits...

What about a closed form for

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Notes