Office hour Pre-A1: Tomorrow 3-6 ... BA3289

Test #1: - Academic Handbook
- Rooms
- office Hours.

CSC236 fall 2016

recurrences...

Danny Heap

heap@cs.toronto.edu / BA4270 (behind elevators)

http://www.cdf.toronto.edu/~csc236h/fall/

416-978-5899

Using Introduction to the Theory of Computation, Chapter 3

Outline

induction on recurrences

Notes

recursively defined function

n	0	1	2	3	4	5	6	7	8	9	10
$f(n)$	2	7	11	25	47	97	191				

define:

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

Write out a few values of $f(n)$. Conjectures?

$f(n) < 2^{n+2}$

$f(n) < 2^{n+2}$: Define $P(n)$: $f(n) < 2^{n+2}$

Proof — complete Induction.

Let n be an arbitrary natural number. Assume $H(n)$: $0 \le i < n$, $P(i)$, that is $f(i) < 2^{i+2}$.

Must show $C(n)$: $P(n)$, that is $f(n) < 2^{n+2}$.

Case $n \in \{0, 1\}$ verify directly

Case $n > 1$ By definition

$$f(n) = 2f(n-2) + f(n-1)$$

$$< 2\,2^{n} + 2^{n+1} \quad \# \text{ By } H(n),\ 0 \le n-2, n-1 < n$$

$$= 2^{n+1} + 2^{n+1} = 2 \cdot 2^{n+1} = 2^{n+2}$$

So $P(n)$, ie. $C(n)$ is confirmed ∎

Recursive definition

Fibonacci sequence

- immortal
- have babies each month except the month they are born.
- Fib. starts 1th month with a breeding pair

0	1	2	3	4	5	6	. . .
0	1	1	2	3	5	8	

This sequence comes up in applied rabbit breeding, the height of AVL trees, and the complexity of Euclid's algorithm for the GCD:

$$\sum_{i=0}^{n} F(i) = F(n+2) - 1$$

$$F(n) = \begin{cases} n & n < 2 \\ F(n-2) + F(n-1) & n \geq 2 \end{cases}$$

What is the sum of n Fibonacci numbers?

n	0	1	2	3	4	5	6	7	8	9	10	
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	
$\sum_{i=0}^{i:n}$	0	1	2	4	7	12	20	33	54	88	143	

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

$\underline{P(n)}: \sum_{i=0}^{n} F(i) = F(n+2) - 1$

Let n be an arbitrary natural number. Assume $H(n)$: $0 \le k < n$, $P(k)$, that is $\sum_{i=0}^{k} F(i) = F(k+2) - 1$

Must show that $C(n)$, ie $P(n)$, follows.

$$\sum_{i=0}^{n} F(i) = \left[\sum_{i=0}^{n-1} F(i)\right] + F(n) \quad \#$$

$$= F(n+1) - 1 + F(n) \quad \# \text{ By } H(n), \text{ since } 0 \le n-1 < n$$

$$= \underline{F(n+2)} - 1 \quad \# \text{ By definition, since } \# \ n+2 > 2$$

So $C(n)$ is verified ■

Try simple Induction

- add n base cases

Fibonacci numbers

What is $\sum_{i=0}^{i=n} F(i)$?

$F(5n)$: is a mult of 5

Fibonacci patterns...

$P(n)$: $F(4 \cdot n)$ is a multiple of 3

what are Fibonacci numbers multiples of?

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	...
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	89	144						

Proof Simple Induction

Base Case $F(4 \cdot 0) = 0 = 3 \times 0$, so
$P(0)$ is verified

Let $n \in \mathbb{N}$. assume $H(n)$: $P(n)$, that $F(4 \cdot n)$ is a multiple of 3.
Must Show $C(n)$ follows, that $P(n+1)$.

$F(4(n+1)) = F(4n+4) = F(4n+2) + F(4n+3)$ # By def $4n+4 > 1$

$= F(4n) + F(4n+1) + F(4n+1) + F(4n+2)$ # By def. since $4n+2, 4n+3 > 1$

$= F(4n) + 2F(4n+1) + F(4n) + F(4n+1)$ # since $4n+2 > 1$

$= 2F(4n) + 3F(4n+1)$.

Let $k \in \mathbb{N}$ s.t. $F(4n) = 3k$ # By $H(n)$

Fibonacci patterns...

what are Fibonacci numbers multiples of?

$F(4(n+1)) = 3(2k + F(4n+1))$ # from prev page.

and $2k + F(4n+1) \in \mathbb{N}$ # $2, k,$ ~~4n~~ $\underline{\underline{F(4n+1)}} \in \mathbb{N}$

So, $P(n+1)$, ie $F(4(n+1))$ is mult of 3.

which establishes $C(n)$. ∎

Closed form for $F(n)$?

No rabbit, no hat

The course notes present a proof by induction that

$$F(n) = \frac{\phi^n - \left(\hat{\phi}\right)^n}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2}, \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

You can, and should, be able to work through the proof. The question remains, how did somebody ever think of ϕ and $\hat{\phi}$?

Closed form

...without rabbit

$$F(n) \stackrel{?}{=} r^n$$

Start with the idea that $F(n)$ seems to increase by something close to a fixed ratio. Let's try calling that r, and r has to satisfy:

$$r^n = r^{n-1} + r^{n-2} \Rightarrow \boxed{r^2 = r + 1}$$

There are two solutions to the quadratic equation: ϕ and $\hat{\phi}$, but what about the $1/\sqrt{5}$ factor?

$$\alpha, \beta \in \mathbb{R}$$

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

$$\alpha\left[\phi^n = \phi^{n-1} + \phi^{n-2}\right] + \beta\left[\hat{\phi}^n = \hat{\phi}^{n-1} + \hat{\phi}^{n-2}\right]$$

If ϕ and $\hat{\phi}$ are solutions, so are linear combinations:

$$\alpha\phi^n + \beta\hat{\phi}^n = \alpha\phi^{n-1} + \beta\hat{\phi}^{n-1} + \alpha\phi^{n-2} + \beta\hat{\phi}^{n-2}$$

$$\alpha\phi^0 + \beta\hat{\phi}^0 = F(0) = 0$$

$$\alpha\phi^1 + \beta\hat{\phi}^1 = F(1) = 1$$

rabbits get hats

$$\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

Match up α and β to solutions:

$$\begin{aligned} \alpha\phi^0 + \beta\hat{\phi}^0 &= 0 \quad \Rightarrow \boxed{\alpha = -\beta} \\ \alpha\phi^1 + \beta\hat{\phi}^1 &= 1 \quad \Rightarrow \alpha(\phi - \hat{\phi}) = 1 \end{aligned}$$

$$\alpha = \frac{1}{\phi - \hat{\phi}}$$

$$\alpha = \frac{1}{\sqrt{5}}$$

$$\beta = -\frac{1}{\sqrt{5}}$$

more rabbits...

$$r^n = 2r^{n-2} + r^{n-1}$$

$$r^2 = 2 + r$$

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2)$$

$r_1 = 2$

$r_2 = -1$

What about a closed form for

$$f(n) = \begin{cases} 2 & n = 0 \\ 7 & n = 1 \\ 2f(n-2) + f(n-1) & n > 1 \end{cases}$$

wishing $f(n) \sim r^n$

$$\boxed{\begin{aligned} \alpha r_1^0 + \beta r_2^0 &= f(0) \\ &= 2 \\ \alpha r_1 + \beta r_2 &= 7 \end{aligned}}$$

Notes