- First tutorial + quiz
- STA 247

office hour: F 4-5

CSC236 fall 2016

complete induction

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use Introduction to the Theory of Computation, Section 1.3

Outline

Principle of complete induction

Examples of complete induction

Complete Induction

another flavour needed

Every natural number greater than 1 has a prime factorization

2 : 2
3 : 3
4 : 2, 2
5 : 5
6 : 2, 3
7 : 7
8 : 2, 2, 2
9 : 3, 3

?

Try some examples

How does the factorization of 8 help with the factorization of 9?

More dominoes

typographic physics
mean

DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO DOMINO

$$(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1)\rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)$$

If all the previous cases always implies the current case
then all cases are true

complete induction outline

inductive step: state inductive hypothesis $H(n)$

can tuck in as special case.

derive conclusion $C(n)$: show that $C(n)$ follows from $H(n)$, indicating where you use $H(n)$ and why that is valid

verify base case(s): verify that the claim is true for any cases not covered in the inductive step

Wait! isn't that the same outline as simple induction?

Yes, we just modify the inductive hypothesis, $H(n)$ so that it assumes the main claim for every natural number from the starting point up to $n - 1$, and the conclusion, $C(n)$, is now the main claim for n.

watch the base cases, part 1

$$f(n) = \begin{cases} 1 & n \leq 1 \\ [f(\lfloor\sqrt{n}\rfloor)]^2 + 2f(\lfloor\sqrt{n}\rfloor) & n > 1 \end{cases}$$

Check a few cases, and make a conjecture

$f(0) - 1$
$f(1) - 1$
$f(2) - 3$
$f(3) - 3$

$f(4) - 15$
$f(5) - 15$
$f(6) - 15$
$\vdots$
$f(9) - 15$

$f(16) - 255$

Conjecture:
$\forall n \in \mathbb{N}$, s.t. $n > 1$
$f(n)$ is a multiple of 3.

For all natural numbers $n > 1$, $f(n)$ is a multiple of $\underline{3}$?

use the complete induction outline

Proof – use complete induction.

Inductive step: Let $n \in \mathbb{N}$, s.t. $n > 1$. Assume $H(n)$:
for $i \in \mathbb{N}$, $2 \leq i < n$, $f(i)$ is a multiple of 3.
show that $H(n) \Rightarrow C(n)$: $f(n)$ is a multiple of 3

Case $n \geq 4$: $f(n) = [f(\lfloor\sqrt{n}\rfloor)]^2 + 2f(\lfloor n \rfloor)$

$2 \leq \lfloor\sqrt{n}\rfloor < n$ # $n \geq 4 \Rightarrow \sqrt{n} \geq 2 > 1$

$\sqrt{n}\cdot\sqrt{n} > \sqrt{n}$, $\sqrt{n} < n$

$f(\lfloor\sqrt{n}\rfloor)$ is a multiple of 3 # by $H(n)$, since

Let $k \in \mathbb{N}$, s.t. $f(\lfloor\sqrt{n}\rfloor) = 3k$ # since $f(\lfloor\sqrt{n}\rfloor)$ mult of 3

$f(n) = (3k)^2 + 2\cdot 3k$
$= 3(3k^2 + 2k)$, where $3k^2 + 2k \in \mathbb{Z}$,
since $3, 2, k \in \mathbb{Z}$
& $\mathbb{Z}$ closed under $+, \times$

$\longrightarrow$

For all natural numbers $n > 1$, $f(n)$ is a multiple of 3?

use the complete induction outline

$f(n)$ is multiple of 3.

Verify base cases: $f(2) = [f(1)]^2 + 2f(1)$

$= 1^2 + 2 = 3 = 3 \times 1$

$f(3) = [f(1)]^2 + 2f(1)$. So

Claim holds for 2 and 3.

zero pair free binary strings, zpfbs...

Denote by *zpfbs*(n) the number of binary strings of length n that contain no pairs of adjacent zeros. What is *zpfbs*(n) for the first few natural numbers n?

$zp(0)$: 1
$zp(1)$: 2
$zp(2)$: 3
$zp(3)$: 5
$zp(4)$: 8
$zp(5)$: 13

what is $zpfbs(n)$?

use the complete induction outline

define $zp(n) = \begin{cases} 1 & n=0 \\ 2 & n=1 \\ zp(n-2)+zp(n-1) & n \geq 2 \end{cases}$

Claim number of zpfbs of length n is $zp(n)$

Proof - complete induction.

Inductive step

Let $n \in \mathbb{N}$. Assume $H(n)$: $\forall i \in \mathbb{N}$, $0 \leq i < n$, $zp(i)$ is the # of zpfbs of length i.

need to show $H(n) \Rightarrow C(n)$: $zp(n)$ # of zpfbs of length n

Case $n \geq 2$ then $0 \leq n-2, n-1 < n$

zpfbs of length n partition into two sets: P_1 - those that end in "1", and P_0 - those that end in 0.

$|P_1| = zp(n-1)$ # by $H(n)$, since $0 \leq n-1 < n$, and these are exactly the zpfbs of length $n-1$ with "1" appended

what is $zpfbs(n)$?

use the complete induction outline

$|P_0| = zp(n-2)$ # by $H(n)$, since $0 \le n-2 < n$
and these zpfbs are exactly the same as those of length $n-2$ with "10" appended.

So, the number of zpfbs of length n is $zp(n-2) + zp(n-1)$ as claimed.

Case (base case) $n \le 1$ the # of zpfbs of length 0 is 1, since the unique empty string has no zeros! and $1 = zp(0)$
the # of zpfbs of length 1 is 2: "1", "0", and $2 = zp(1)$

In all cases $C(n)$ follows from $H(n)$ ∎

After a certain natural number n, every postage can be made up by combining 3− and 5− cent stamps

what is the "certain natural number"?

0 ✓
1 ✗
2 ✗
3 ✓
4 ✗
5 ✓
6 ✓
7 ✗
8 ✓
9 ✓
10 ✓
11 ✓

After a certain natural number n, every postage can be made up by combining 3− and 5− cent stamps

use the complete induction outline

Proof by CI

Let $n \in \mathbb{N}$, s.t. $n \geq 8$. Assume $H(n)$: $\forall i \in \mathbb{N}$, $8 \leq i < n$, postage of i¢ can be made with 3¢ and 5¢ stamps.

Show $H(n) \Rightarrow C(n)$: postage of n¢ can be made with 3¢ and 5¢ stamps.

exercise

After a certain natural number n, every postage can be made up by combining $3-$ and $5-$ cent stamps

use the complete induction outline

Proof by Complete induction

Let $n \in \mathbb{N}$

Every natural number greater than 1 has a prime factorization

use the complete induction outline

prime factorization: representation as product of 1 or more primes

Every natural number greater than 1 has a prime factorization

use the complete induction outline

Proof by CI

Let $n \in \mathbb{N}$ s.t. $n \geq 2$. assume $H(n)$: $\forall i \in \mathbb{N}$ $2 \leq i < n$, i has a prime factorization.

Show $H(n) \Rightarrow C(n)$: n has a prime factorization

Case n is prime - then n is its own prime factorization

Case n is composite n has integer factors f_1, f_2, $1 < f_1, f_2 < n$ and $f_1 \cdot f_2 = n$

f_1 and f_2 have prime factorizations # by $H(n)$, # $2 \leq f_1, f_2 < n$

let $f_1 = p_1 \cdots p_j$, p_s are prime

$f_2 = q_1 \cdots q_k$, q_s are prime

$n = f_1 f_2 = p_1 \cdots p_j \, q_1 \cdots q_k$ has PF

So $C(n)$ follows from $H(n)$.