CSC236 *Intro. to* the Theory of Computation

Lecture 8: correctness proof of iterative programs

Amir H. Chinaei, Fall 2016

Office Hours: W 2-4 BA4222

ahchinaei@cs.toronto.edu
http://www.cs.toronto.edu/~ahchinaei/

Course page:
http://www.cdf.toronto.edu/~csc236h/fall/index.html

Section page:
http://www.cdf.toronto.edu/~csc236h/fall/amir_lectures.html

review

- Last lecture
  - correctness proof of **recursive** programs
    - *based on the program specification*
      - *i.e., pre- & post- conditions*
    - *normally using induction*
      - *proof is parallel to the code*
- this week
  - correctness proof of **iterative** programs
    - *based on the program specification*
      - *i.e., pre- & post- conditions*

Example 76: x^n

```
def power(x, n):
    r = 1
    c = 0
    while c < n:
        r = r * x
        c = c + 1
    return r
```

- How to start the proof, formally?
  - recall, we need to show:
    preconditions ⇒ postconditions

Example 76: pre- post- conditions

- power(x, n):
  - **preconditions:**
    - $0 \leq n \in \mathbb{N}$
    - $x \in \mathbb{R}$
  - **postconditions:**
    - power(x, n) terminates and returns x^n

Example 76: correctness of power, x^n:

- We want to show: preconditions ⇒ postconditions

  $P(n)$: if $n \in \mathbb{N}, x \in \mathbb{R}$, then *power(x,n)* terminates and returns x^n.

- **Partial correctness:**

  $P'(n)$: if $n \in \mathbb{N}, x \in \mathbb{R}$, and *power(x,n)* terminates, then it returns x^n.

- To prove ***partial correctness*** of iterative algorithms, we use
  - **loop invariant:** an assertion (predicate) that must be true before and after each iteration of the loop

Example 76: *loop invariant*

```
def power(x, n):
    r = 1
    c = 0
    while c < n:
        r = r * x
        c = c + 1
    return r
```

- loop invariant:
  - $LI(c, r): 0 \leq c \leq n$ *and* $r = x^c$

$LI(c, r)$: $0 \le c \le n$ and $r = x^c$

recipe

- show the LI holds before the loop starts
  - show $LI(c_0, r_0)$ holds
- show the LI holds at each iteration of the loop
  - show $LI(c_k, r_k) \to LI(c_{k+1}, r_{k+1})$
- show the LI holds after the loop exits
  - by showing $LI(c_n, r_n)$ holds,
    - conclude the postcondition is met.

$LI(c, r)$: $0 \le c \le n$ and $r = x^c$

Example 76: *partial correctness*

```
def power(x, n):
    r = 1
    c = 0

    while c < n:
        r = r * x
        c = c + 1

    return r
```


$LI(c, r)$: $0 \le c \le n$ and $r = x^c$

Example 76: *partial correctness*

```
def power(x, n):
    r = 1
    c = 0

    while c < n:
        r = r * x
        c = c + 1

    return r
```


$LI(c, r)$: $0 \le c \le n$ and $r = x^c$

Example 76: *partial correctness*

```
def power(x, n):
    r = 1
    c = 0

    while c < n:
        r = r * x
        c = c + 1

    return r
```


an important task remains:

- Termination of `power`
  - To prove ***termination*** of iterative algorithms, we use
    - `loop variant`: a number, k_c, associated with each iteration c of the loop, and we show
      - $k_c \in \mathbb{N}$
      - k_c is decreasing

Example 76: *loop variant*

```
def power(x, n):
    r = 1
    c = 0
    while c < n:
        r = r * x
        c = c + 1
    return r
```

- loop variant:
  - $k_c = n - c$

Example 76: *terminations*

$k_c = n - c$

```
def power(x, n):
    r = 1
    c = 0

    while c < n:
        r = r * x
        c = c + 1

    return r
```


notes

Example 77: *iterative binSearch*

```
def iteBinSearch(x, A):
    # Precondition: A is a sorted array, and
    #               A is not empty.
    # Postcondition: Return an integer p such that 0<= p <= length(A)
    # and A[p] = x, if such a p exists; otherwise return -1.

    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2    # midpoint
        if x <= A[m]:
            e = m
        else:
            b = m + 1
    if A[b] == x:
        return b
    else:
        return -1
```


Example 77: pre- post- conditions

- `iteBinSearch(x, A)`:
  - **preconditions:**
    - A is not empty
    - elements of A are sorted non-decreasingly
    - elements of A and x are comparable
  - **postconditions:**
    - `iteBinSearch(x, A)` terminates and returns p such that $0 \le p \le Length(A) - 1$ and $x = A[p]$ if such a p exists;
    - otherwise it terminates and returns -1.

Example 77: correctness of `iteBinSearch`:

- We want to show: preconditions $\Rightarrow$ postconditions

  $\boldsymbol{P(n)}$: **if** A is not empty and non-decreasing and $n = \text{Length(A)} > 1$, and x is comparable to elements of A, **then** *`iteBinSearch(x,A)`* terminates and returns p such that $0 \le p \le Length(A) - 1$ and $x = A[p]$ if such a p exists; otherwise it terminates and returns -1.

- **Part A) Partial correctness:**

  $\boldsymbol{P'(n)}$: **if** A is not empty and non-decreasing and $\text{n} = \text{Length(A)} > 1$, and x is comparable to elements of A and *`iteBinSearch(x,A)`* terminates, **then** it returns p such that $0 \le p \le Length(A) - 1$ and $x = A[p]$ if such a p exists; otherwise it returns -1.

- **Part B) Termination:**

  $\boldsymbol{P''(n)}$: **if** A is not empty and non-decreasing and $\text{n} = \text{Length(A)} > 1$, and x is comparable to elements of A, **then** *`iteBinSearch(x,A)`* terminates.

Example 77: *loop invariant*

```
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1
    while b != e:
        m = (b + e) // 2    # midpoint
        if x <= A[m]: e = m
        else: b = m + 1
    if A[b] == x: return b
    else:return -1
```

- loop invariant:
  - ...

recipe

- show the LI holds before the loop starts
  - show $LI(b_0, e_0)$ holds
- show the LI holds at each iteration of the loop
  - show $LI(b_k, e_k) \rightarrow LI(b_{k+1}, e_{k+1})$
- show the LI holds after the loop exits
  - and from there, conclude the postcondition is met.

We use induction to show the LI holds for all i's.

Example 77: *partial correctness*

$P(i)$: if the loop iterates at least i times, $LI(b_i, e_i)$ holds.

```
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2
        if x <= A[m]:
            e = m
        else:
            b = m + 1

    if A[b] == x:
        return b
    else:
        return -1
```


Example 77: *partial correctness*

$P(i)$: if the loop iterates at least i times, $LI(b_i, e_i)$ holds.

```
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2
        if x <= A[m]:
            e = m
        else:
            b = m + 1

    if A[b] == x:
        return b
    else:
        return -1
```


$LI(b_i, e_i)$: $0 \le b_i \le e_i \le n-1$ and if x is in A, it's in $A[b_i..e_i]$

Example 77: *partial correctness*

$P(i)$: if the loop iterates at least i times, $LI(b_i, e_i)$ holds.

```
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2
        if x <= A[m]:
            e = m
        else:
            b = m + 1

    if A[b] == x:
        return b
    else:
        return -1
```


an important task remains:

- Termination of `iteBinSearch`
  - To prove ***termination*** of iterative algorithms, we use
    - **loop variant:** a number, k_i, associated with each iteration i of the loop, and we show
      - $k_i \in \mathbb{N}$
      - k_i is decreasing

Example 77: *loop variant*

- loop variant:

```
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2
        if x <= A[m]:
            e = m
        else:
            b = m + 1

    if A[b] == x:
        return b
    else:
        return -1
```

Example 77: *terminations*

```python
def iteBinSearch(x, A):
    b = 0
    e = len(A)-1

    while b != e:
        m = (b + e) // 2
        if x <= A[m]:
            e = m
        else:
            b = m + 1

    if A[b] == x:
        return b
    else:
        return -1
```


notes