CSC236 *Intro. to* the Theory of Computation

Lecture 2: *Strong Induction*

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Course page:
http://www.cdf.toronto.edu/~csc236h/fall/index.html

Section page:
http://www.cdf.toronto.edu/~csc236h/fall/amir_lectures.html

recall

- use all resources available to you
  - before it becomes too late!
- what resources?
  - office Hours:
    - **M 2-3:30 in PT286C, W 2-4 BA4222, F 3:30-4:30 BA4270**
  - the course page and our section page
  - the CS Help Centre
  - the course forum
  - study groups and Peer Instruction
  - email ahchinaei @ cs.torotno.edu

review

- Last week
  - **Simple Induction**
    - AKA: **Mathematical Induction** or **Principle of Mathematical Induction**
  - 17 examples
- This week
  - **Strong Induction**
    - AKA: **Complete Induction** or **Second Principle of Mathematical Induction**
- Next week
  - **Structural Induction**
  - **Well Ordering**

review

- Simple Induction
  - it's a rule of inference:

$$\begin{array}{ll} P(b) & \\ P(k) \rightarrow P(k+1) & \forall k \geq b \in \mathbb{N} \\ \hline P(n) & \forall n \geq b \in \mathbb{N} \end{array}$$

  - *after all,*
    - *To show that all domino pieces fall over, we should show that*

1) there is a starting point, i.e., *P(b) holds*

and 2) all pieces are set in a well order such that falling of piece k implies falling of piece k+1

i.e., *and P(k) → P(k+1) holds too.*

yet another example

- **Example 19.** a student who went to office hour 01 has provided the following claim and proof. Is it valid?
- **Conjecture:** doubts in csc236 can be clarified by further discussion each week (e.g., going to the week's office hours).
  - Let $P(n)$ denotes d_n –read doubts of week n–can be clarified by further discussion.
  - Proof by simple induction.
  - **Basis step**: $P(1)$ holds as new doubts were clarified in office hours of week 01.
  - **Inductive step**: we assume that doubts of week k can be clarified by further discussion in that week. We need to show that doubts of week k+1 can be clarified too. There are two cases: doubts of week k+1 are either from week k (that can be clarified by further discussion, based on the **I.H.**) or they are new doubts (basis step). This completes the inductive step.
  - Therefore, by simple induction, all doubts in csc236 can be clarified by further discussion.

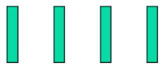
□

proof by strong induction

- **recipe:**
  - to prove that $P(n)$ is true for all natural numbers n, we should demonstrate these steps:
    - ***Proof method:*** "strong induction"
    - ***Basis step***: show that $P(n)$ is true for some starting point(s), usually 0 or 1 but not always
    - ***Inductive step***: show that $P(k) \rightarrow P(k+1)$ is true for all natural numbers k *greater than the starting point.*
      - to complete the inductive step, assume H —i.e., $P(i)$— holds ***for all i's where b≤i≤k*** for an arbitrary natural number k, show that C must be true.

revisit: proof by simple induction

fall of the *k+1*th piece is implied by fall of the previous piece, *k*

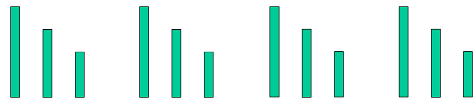


In the Inductive Step, we show that:

$\boldsymbol{f(d_k)}$ implies $\boldsymbol{f(d_{k+1})}$

proof by strong induction

fall of the *k+1*th piece is implied by fall of all previous pieces, *b..k*



In the Inductive Step, we show that:

$\boldsymbol{f(d_b) \wedge f(d_{b+1}) \wedge .. \wedge f(d_k)}$ implies $\boldsymbol{f(d_{k+1})}$

Example 20:

P(*n*): *n* can be written as product of prime numbers; $\forall n \geq 2 \in \mathbb{N}$.

scratch work

n	products	P(n)
...	...	...

Example 20:

P(*n*): *n* can be written as product of primes; $\forall n \geq 2 \in \mathbb{N}$.

Proof by Strong Induction

Basis step: *P*(2) holds because 2 is prime.

Inductive step:

Inductive Hypothesis: Assume *P*(*i*) holds for all $i \in \mathbb{N}$ where $2 \leq i \leq k$ for any arbitrary fixed $k \in \mathbb{N}$, *i.e.*, we assume all numbers less than or equal *k* can be written as product of primes.

We need to show that *P*(*k*+1) holds too, *i.e.*, *k+1* can be written as product of primes too.

There are two cases: if *k*+1 is prime, we are done as P(*k*+1) holds.

If *k*+1 is not prime, it's composite and can be written as:

Example 20:

P(*n*): *n* can be written as product of primes; $\forall n \geq 2 \in \mathbb{N}$.

If *k*+1 is not prime, it's composite and can be written as:

$k+1 = m.n$ where $m,n \in \mathbb{N}$ and $2 \leq m,n < k+1$, *i.e.*, $2 \leq m,n \leq k$

By the Inductive hypothesis, *m* and *n* each can be written by product of primes. Therefore, *k*+1 is a product of primes, *i.e.*, *P*(*k*+1) holds too.

This completes the inductive step.

Therefore, by strong induction, it proves that $\forall n \geq 2 \in \mathbb{N}$, *n* can be written as product of primes.

□

Example 21:

P(*n*): any postage *n* that is 18 cents or more can exactly be stamped using just 4-cent and 7-cent stamps.

scratch work

n	stamps	P(n)
18	1*4+2*7	√
...	...	...

Example 21:

P(n): n can be written as $a*4 + b*7$; $\forall n \geq 18 \in \mathbb{N}$.

Proof by Strong Induction

Strong Induction 2-13

Example 21:

P(n): n can be written as $a*4 + b*7$; $\forall n \geq 18 \in \mathbb{N}$.

Strong Induction 2-14

Example 22:

Make a conjecture to specify the minimum number of breaks to break a chocolate bar to all chocolate squares. Proof your claim.

scratch work

Strong Induction 2-15

Example 22:

Strong Induction 2-16

Example 22:

Strong Induction 2-17

Example 23:

Let $f(n) = \begin{cases} 2 & n = 1 \\ f^2(\lfloor\sqrt{n}\rfloor) + 2f(\sqrt{n}) & n > 1 \end{cases}$, prove $f(n)$ is a multiple of 8.

scratch work

Strong Induction 2-18

Example 23:

Example 23:

Example 24:

Prove that every simple polygon with *n* sides can be composed of *n*-2 triangles.

scratch work

Example 24:

P(*n*): An *n-sided* polygon can be triangulated to *n*-2 triangles; $\forall n \geq 18 \in \mathbb{N}$.

Example 24:

P(*n*): An *n-sided* polygon can be triangulated to *n*-2 triangles; $\forall n \geq 18 \in \mathbb{N}$.

strong induction recipe (revisited)

0. write out the claim as: "**Let *P(n)* denote the claim in terms of *n***"
   follow next steps to show that *P*(*n*) holds $\forall n \geq b \in \mathbb{N}$, where *b* is staring point(s)
1. write out "***Proof method:* strong induction**"
2. write out "***Basis step:*"** followed by reasoning that *P*(*b*) is true
3. write out "***Inductive step:***"
   - 3.1. write out "**Inductive hypothesis:** we assume *P*(*i*) is true $\forall i$, $b \leq i \leq k$" where *P*(*i*) is the claim in terms of *i*
   - 3.2. reason that *P*(*k*+1) is true
     - **note 1:** in your reasoning here, you must use the inductive hypothesis
     - **note 2:** be sure your reasoning is true for any $k \geq b$, including *k*=*b*
   - 3.3. write out "**This completes the inductive step**"
4. write out "***This proves P(n) is true for*** $\forall n \geq b \in \mathbb{N}$" where *P*(*n*) is the claim in terms of *n*
5. Indicate end of proof by "□".

Notes:

Strong Induction 2-25

Notes:

Strong Induction 2-26