The aim of this assignment is to give you some practice with various forms of induction. For each question below you will present a proof by induction. For full marks you will need to make it clear to the reader that the base case(s) is/are verified, that the inductive step follows for each element of the domain (typically the natural numbers), where the inductive hypothesis is used and that it is used in a valid case.

Your assignment must be typed to produce a PDF document a1.pdf (hand-written submissions are not acceptable). You may work on the assignment in groups of 1, 2, or 3, and submit a single assignment for the entire group on MarkUs.

1. Consider the Fibonacci-esque function $g$:

$$
g(n) = \begin{cases} 
1, & \text{if } n = 0 \\
3, & \text{if } n = 1 \\
g(n-2) + g(n-1), & \text{if } n > 1 
\end{cases}
$$

Use complete induction to prove that if $n$ is a natural number greater than 1, then $2^{n/2} \leq g(n) \leq 2^n$. You may not derive or use a closed-form for $g(n)$ in your proof.

2. Suppose $B$ is a set of binary strings where each binary string is of length $n$. $n$ is positive (greater than 0), and no two strings in $B$ differ in fewer than 2 positions. Use simple induction to prove that $B$ has no more than $2^n$ elements.

3. Define $T$ as the smallest set of strings such that:

(a) "b" $\in T$

(b) If $t_1, t_2 \in T$, then $t_1 + "ene" + t_2 \in T$, where the + operator is string concatenation.

Use structural induction to prove that if $t \in T$ has $n$ "b" characters, then $t$ has $2n - 2"e"$ characters.

4. On page 79 of the Course Notes the quantity $\phi = (1 + \sqrt{5})/2$ is shown to be closely related to the Fibonacci function. You may assume that $1.61803 < \phi < 1.61804$. Complete the steps below to show that $\phi$ is irrational.

(a) Show that $\phi(\phi - 1) = 1$.

(b) Rewrite the equation in the previous step so that you have $\phi$ on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, $+$ or $-$, and $\phi$. There are two different fractions, corresponding to the two different factors in the original equation’s left-hand side. Keep both fractions around for future consideration.
(c) Assume, for a moment, that there are natural numbers \( m \) and \( n \) such that \( \phi = n/m \). Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, + or -, \( m \) and \( n \).

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains \( m \). Use the Principle of Well-Ordering, plus one of the two expressions for \( \phi \) from the previous step to derive a contradiction.

(e) Combine your assumption and contradiction from the previous step into a proof that \( \phi \) cannot be the ratio of two natural numbers. Extend this to a proof that \( \phi \) is irrational.

5. Consider the function \( f \), where \( 3 \div 2 = 1 \) (integer division, like \( 3/2 \) in Python):

\[
f(n) = \begin{cases} 
1 & \text{if } n = 0 \\
\frac{n^2}{3} + 3f(n) & \text{if } n > 0 
\end{cases}
\]

Use complete induction to prove that for every natural number \( n \) greater than 2, \( f(n) \) is a multiple of 7. NB: Think carefully about which natural numbers you are justified in using the inductive hypothesis for.