1. Write a detailed structured proof that $5n^4 - 3n^2 + 1 \in \mathcal{O}(6n^5 - 4n^3 + 2n)$.

Proof outline: By definition of "$\mathcal{O}$", we have to show

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 5n^4 - 3n^2 + 1 \leqslant c(6n^5 - 4n^3 + 2n).$$

This can be done using the following proof structure.
Let $c' = \ldots$ Then $c' \in \mathbb{R}^+$.
Let $B' = \ldots$ Then $B' \in \mathbb{N}$.
Assume $n \in \mathbb{N}$ and $n \geqslant B'$.
...show that $5n^4 - 3n^2 + 1 \leqslant c'(6n^5 - 4n^3 + 2n)$...
Then $\forall n \in \mathbb{N}, n \geqslant B' \Rightarrow 5n^4 - 3n^2 + 1 \leqslant c'(6n^5 - 4n^3 + 2n)$.
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 5n^4 - 3n^2 + 1 \leqslant c(6n^5 - 4n^3 + 2n)$.

Scratch work: Working "forward" from the left-hand side, we get:

$$\begin{aligned}
5n^4 - 3n^2 + 1 &\leqslant 5n^4 + 1 \\
&\leqslant 5n^4 + n^4 && \text{if } n \geqslant 1 \\
&\leqslant 6n^4 \\
&\leqslant n \cdot n^4 && \text{if } n \geqslant 6 \\
&\leqslant n^5
\end{aligned}$$

(Note that there are other inequalities we could have reached, *e.g.*, $6n^4 \leqslant 6n^5$ for all $n \geqslant 1$.)
Working "backward" from the right-hand side, we get:

$$\begin{aligned}
6n^5 - 4n^3 + 2n &\geqslant 6n^5 - 4n^3 \\
&\geqslant 6n^5 - 4n^5 && \text{because } -n^3 \geqslant -n^5 \\
&\geqslant 2n^5 \\
&\geqslant n^5
\end{aligned}$$

Since both chains of inequalities connect, we are done: we can pick $B = 6$ (because we require $n \geqslant 6$ in our first chain) and $c = 1$.

Proof: (This is the actual final "solution". We skip the formal introduction of B' and c' and instead, simply use their values directly. This style of proof is fine, and it is a little less verbose than using "B'" and "c'" throughout the argument.)
Assume $n \in \mathbb{N}$ and $n \geqslant 6$.

$$\begin{aligned}
\text{Then, } 5n^4 - 3n^2 + 1 &\leqslant 5n^4 + 1 \\
&\leqslant 5n^4 + n^4 && \text{since } n \geqslant 6 > 1 \\
&\leqslant 6n^4 \\
&\leqslant n \cdot n^4 && \text{since } n \geqslant 6 \\
&\leqslant 2n^5 \\
&\leqslant 6n^5 - 4n^5 \\
&\leqslant 6n^5 - 4n^3 && \text{since } -n^5 \leqslant -n^3 \\
&\leqslant 6n^5 - 4n^3 + 2n.
\end{aligned}$$

Then $\forall n \in \mathbb{N}, n \geqslant 6 \Rightarrow 5n^4 - 3n^2 + 1 \leqslant 6n^5 - 4n^3 + 2n$.
Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geqslant B \Rightarrow 5n^4 - 3n^2 + 1 \leqslant c(6n^5 - 4n^3 + 2n)$. # pick $B = 6$ and $c = 1$
Then $5n^4 - 3n^2 + 1 \in \mathcal{O}(6n^5 - 4n^3 + 2n)$, by definition.

2. Write a detailed structured proof that $6n^5 - 4n^3 + 2n \notin \mathcal{O}(5n^4 - 3n^2 + 1)$.

Proof outline: From the *negated* definition of "$\mathcal{O}$", we have to prove

$$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant B \wedge 6n^5 - 4n^3 + 2n > c(5n^4 - 3n^2 + 1)$$

This can be done with the following proof structure.
Assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n_0 = \ldots$ # an expression containing B and c
...show that $n_0 \in \mathbb{N}$...
...show that $n_0 \geqslant B$...
...show that $6n_0^5 - 4n_0^3 + 2n_0 > c(5n_0^4 - 3n_0^2 + 1)$...
Then, $\exists n \in \mathbb{N}, n \geqslant B \wedge 6n^5 - 4n^3 + 2n > c(5n^4 - 3n^2 + 1)$.
Then, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant B \wedge 6n^5 - 4n^3 + 2n > c(5n^4 - 3n^2 + 1)$.

Scratch work: The property of n_0 that will be most difficult to show is $6n_0^5 - 4n_0^3 + 2n_0 > c(5n_0^4 - 3n_0^2 + 1)$, so we focus on it first. It is tempting to try to solve for n_0 — and if the expression were simpler, this would yield an appropriate value. But it will be complicated in this case, and it is not necessary. Remember that, intuitively, we are simply trying to prove that $6n^5 - 4n^3 + 2n$ is larger than $5n^4 - 3n^2 + 1$ by more than a constant factor.

Working "forward" from the left-hand side, we get:

$$\begin{aligned} 6n^5 - 4n^3 + 2n &> 6n^5 - 4n^3 && \text{if } n \geqslant 1 \\ &\geqslant 6n^5 - 4n^5 && \text{if } n \geqslant 1 \\ &= 2n^5 \end{aligned}$$

Working "backward" from the right-hand side, we get:

$$\begin{aligned} 5n^4 - 3n^2 + 1 &< 5n^4 + 1 && \text{if } n \geqslant 1 \\ &\leqslant 6n^4 && \text{if } n \geqslant 1 \end{aligned}$$

Now, we want $2n^5 > c(6n^4)$, *i.e.*, $n^5 > 3cn^4$. This will be true as long as $n > 3c$. Since $c \in \mathbb{R}^+$, to ensure $n \in \mathbb{N}$, we can pick any value $n \geqslant \lceil 3c \rceil + 1$. This guarantees $n \geqslant 1$, which is needed for the inequalities above to hold. Finally, we also need $n \geqslant B$, which can be achieved simply by picking $n = B + \lceil 3c \rceil + 1$.

Proof:

Assume $c \in \mathbb{R}^+$ and $B \in \mathbb{N}$.
Let $n_0 = B + \lceil 3c \rceil + 1$.
Then $n_0 \in \mathbb{N}$ because $B \in \mathbb{N}$ and $\lceil 3c \rceil \in \mathbb{N}$ for all $c \in \mathbb{R}^+$.
Then $n_0 \geqslant B$ (in fact, $n_0 > B$).
Then

$$\begin{aligned} 6n_0^5 - 4n_0^3 + 2n_0 &> 6n_0^5 - 4n_0^3 && \text{since } n_0 \geqslant 1 \\ &\geqslant 6n_0^5 - 4n_0^5 && \text{since } n_0 \geqslant 1 \\ &= 2n_0^5 \\ &= n_0(2n_0^4) \\ &> 3c(2n_0^4) && \text{since } n_0 > 3c \\ &= c(6n_0^4) \\ &\geqslant c(5n_0^4 + 1) && \text{since } n_0 \geqslant 1 \\ &> c(5n_0^4 - 3n_0^2 + 1). \end{aligned}$$

Then, $\exists n \in \mathbb{N}, n \geqslant B \wedge 6n^5 - 4n^3 + 2n > c(5n^4 - 3n^2 + 1)$.

Then, $\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geqslant B \wedge 6n^5 - 4n^3 + 2n > c(5n^4 - 3n^2 + 1)$.
Then $6n^5 - 4n^3 + 2n \notin \mathcal{O}(5n^4 - 3n^2 + 1)$, by definition.