Prove or disprove each of the following statements. Write detailed proof structures and justify your work.

1. For all natural numbers n, if there is a natural number k such that $n = 3k + 1$, then there is a natural number j such that $n^2 = 3j + 1$.

   (Optional), write the statement symbolically, to help understand the structure:

   $$\forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 3k + 1) \Rightarrow \left(\exists j \in \mathbb{N}, n^2 = 3j + 1\right)$$

   Now Try a direct proof:
   Assume n is a generic natural number # in order to introduce $\forall$
   Assume $\exists k \in \mathbb{N}, n = 3k + 1$ # in order to introduce $\Rightarrow$
   Then $n^2 = (3k + 1)^2$ # sub in k from assumption
   Then $n^2 = 3(3k^2 + 2k) + 1$ # expand and factor 3 out
   Then $\exists j \in \mathbb{N}, n^2 = 3j + 1$ # pick $j = (3k^2 + 2) \in \mathbb{N}$
   # since $3, k, 2 \in \mathbb{N}$, which is closed under $+$ and $\times$
   Then $\exists k \in \mathbb{N}, n = 3k + 1 \Rightarrow \exists j \in \mathbb{N}, n^2 = 3j + 1$ # introduced $\Rightarrow$
   Conclude $\forall n \in \mathbb{N}(\exists k \in \mathbb{N}, n = 3k + 1) \Rightarrow (\exists j \in \mathbb{N}, n^2 = 3j + 1)$ # introduced $\forall$

2. For all real numbers r, s, if r and s are both positive, then $\sqrt{r} + \sqrt{s} = \sqrt{r + s}$.

   It seems too good to be true, so try a disproof. First, write the negation symbolically to understand the structure:

   $$\exists r, s \in \mathbb{R}, (r > 0 \wedge s > 0) \wedge \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$$

   Pick $r = 1, s = 1$ # the first, and easiest, reals to work with to introduce $\exists$
   Then $r, s \in \mathbb{R}$ # $r = s = 1 \in \mathbb{R}$
   Then $r > 0 \wedge s > 0$ # $1 > 0$
   Then $\sqrt{r} + \sqrt{s} = \sqrt{1} + \sqrt{1} = 1 + 1 = 2 \neq \sqrt{2} = \sqrt{1 + 1} = \sqrt{r + s}$ # substitute $r = s = 1$ and arithmetic
   Then $\exists r, s \in \mathbb{R}, (r > 0 \wedge s > 0) \wedge \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$ # introduced $\exists$

3. For all real numbers r, s, if r and s are both positive, then $\sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$.

   First, write the statement symbolically:

   $$\forall r \in \mathbb{R}^+, \forall s \in \mathbb{R}^+, r > 0 \wedge s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r + s}$$

   Second, try a direct proof:
   Assume $r \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.
   Assume $r > 0$ and $s > 0$.
   Then, $\sqrt{r} + \sqrt{s} = \ldots$ *No obvious way to continue.*
   Next, try an indirect proof:
   Assume $r \in \mathbb{R}^+$ and $s \in \mathbb{R}^+$.
   Assume $\sqrt{r} + \sqrt{s} = \sqrt{r + s}$.
   Then, $(\sqrt{r} + \sqrt{s})^2 = (\sqrt{r + s})^2$. # square both sides
   Then, $(\sqrt{r})^2 + 2\sqrt{r}\sqrt{s} + (\sqrt{s})^2 = r + s$. # expand both sides
   Then, $2\sqrt{rs} = 0$. # subtract $r + s$ from both sides
   Then, $rs = 0$. # divide by 2 and square both sides

Then, $r = 0 \lor s = 0$.
 # Now, do a sub-proof by cases.
Assume $r = 0$.
Then, $r \not> 0$.
Then, $r \not> 0 \lor s \not> 0$.
Then, $\neg(r > 0 \land s > 0)$.
Assume $s = 0$.
Then, $s \not> 0$.
Then, $r \not> 0 \lor s \not> 0$.
Then, $\neg(r > 0 \land s > 0)$.
In either case, $\neg(r > 0 \land s > 0)$.
Then, $r > 0 \land s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$. # introduced contrapositive
Then, $\forall r \in \mathbb{R}^+, \forall s \in \mathbb{R}^+, r > 0 \land s > 0 \Rightarrow \sqrt{r} + \sqrt{s} \neq \sqrt{r+s}$.

4. For all real numbers x and y, $x^4 + x^3y - xy^3 - y^4 = 0$ exactly when $x = \pm y$.

First, write the statement symbolically (be careful to handle "±" correctly):

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \vee x = -y)$$

Second, start the proof structure for the universal quantifiers:

Assume $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

  # *To prove an equivalence, we prove the implication in each direction.*

  First assume $x^4 + x^3y - xy^3 - y^4 = 0$.

    Then, $x^3(x + y) - y^3(x + y) = 0$. # factor out the expression

    Then, $(x^3 - y^3)(x + y) = 0$. # factor out the expression

    Then, $x^3 - y^3 = 0 \vee x + y = 0$. # $ab = 0 \Leftrightarrow a = 0 \vee b = 0$

    # *Now, do a sub-proof by cases.*

    Assume $x^3 - y^3 = 0$.

      Then, $x^3 = y^3$ # add y^3 to both sides

      Then, $x = y$ # take cube roots on both sides

      Then, $x = y \vee x = -y$ # introduce $\vee$

    Assume $x + y = 0$.

      Then, $x = -y$ # subtract y from both sides

      Then, $x = y \vee x = -y$ # introduce $\vee$

    In either case, $x = y \vee x = -y$.

  Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Rightarrow x = \pm y$.

  Next assume $x = \pm y$.

    Then, $x = y \vee x = -y$. # expand "±"

    # *Now, do a sub-proof by cases.*

    Assume $x = y$.

      Then, $x^3 = y^3$. # cube both sides

      Then, $x^3 - y^3 = 0$. # subtract y^3 from both sides

      Then, $(x^3 - y^3)(x + y) = 0$. # multiply both sides by $(x + y)$

      Then, $x^4 + x^3y - xy^3 - y^4 = 0$. # expand

    Assume $x = -y$.

      Then, $x + y = 0$. # add y to both sides

      Then, $(x^3 - y^3)(x + y) = 0$. # multiply both sides by $(x^3 - y^3)$

      Then, $x^4 + x^3y - xy^3 - y^4 = 0$. # expand

    In both cases, $x^4 + x^3y - xy^3 - y^4 = 0$.

  Then, $x = \pm y \Rightarrow x^4 + x^3y - xy^3 - y^4 = 0$.

  Then, $x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow x = \pm y$. # introduce $\Leftrightarrow$

Then, $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^4 + x^3y - xy^3 - y^4 = 0 \Leftrightarrow (x = y \vee x = -y)$.

(Notice how the detailed proof structure makes it easy to keep track of assumptions, and cases and sub-cases, and to know exactly when we are done.)