1. Write detailed proof *structures* for each of the following statements. **Don't write complete proofs** — for now, focus on the proof structure only and leave out *all* of the actual "content".

(a) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$

Assume $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. # domain assumption
  Assume $x \leqslant y$. # antecedent
    Let $z' = \ldots$ # some value that depends on $x$ and $y$
    [... proof of $z' \in \mathbb{Z}$...]
    Then, $z' \in \mathbb{Z}$.
    [... proof of $x \leqslant z' \leqslant y$...]
    Then, $x \leqslant z' \leqslant y$.
    Then, $\exists z \in \mathbb{Z}, x \leqslant z \leqslant y$. # introduce $\exists$
  Then, $x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$. # introduce $\Rightarrow$
Then, $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$. # introduce $\forall$

(b) $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$

Assume $x \in \mathbb{Z}$. # domain assumption
  Assume $\exists y \in \mathbb{Z}, x = 3y + 1$. # antecedent
    Let $y_0$ be such that $x = 3y_0 + 1$. # by assumption
    Let $y' = \ldots$ # some value that depends on $x$ and $y_0$
    [... proof of $y' \in \mathbb{Z}$...]
    Then, $y' \in \mathbb{Z}$.
    [... proof of $x^2 = 3y' + 1$...]
    Then, $x^2 = 3y' + 1$.
    Then, $\exists y \in \mathbb{Z}, x^2 = 3y + 1$. # introduce $\exists$
  Then, $(\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$. # introduce $\Rightarrow$
Then, $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$. # introduce $\forall$

(c) $\neg \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x \wedge a_y > a_x$

First, we work the negation into the statement.

$$
\begin{aligned}
\neg \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x \wedge a_y > a_x &\iff \exists x \in \mathbb{N}, \neg \exists y \in \mathbb{N}, y > x \wedge a_y > a_x \\
&\iff \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, \neg(y > x \wedge a_y > a_x) \\
&\iff \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow \neg(a_y > a_x) \\
&\iff \exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow a_y \leqslant a_x
\end{aligned}
$$

Then, we can write down the proof structure.
Let $x' = \ldots$
[... proof of $x' \in \mathbb{N}$...]
Then, $x' \in \mathbb{N}$.
Assume $y \in \mathbb{N}$. # domain assumption
  Assume $y > x'$. # antecedent
    [... proof of $a_y \leqslant a_{x'}$...]
    Then, $a_y \leqslant a_{x'}$.
  Then, $y > x' \Rightarrow a_y \leqslant a_{x'}$. # introduce $\Rightarrow$
Then, $\forall y \in \mathbb{N}, y > x' \Rightarrow a_y \leqslant a_{x'}$. # introduce $\forall$
Then, $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow a_y \leqslant a_x$. # introduce $\exists$

2. Now, complete the proofs of each statement from the previous question.

(a) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$

Assume $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$.
    Assume $x \leqslant y$.
        Let $z' = y$.
        Then, $z' \in \mathbb{Z}$. # because $y \in \mathbb{Z}$
        Then, $x \leqslant z'$ and $z' \leqslant y$. # because $x \leqslant y$
        Then, $\exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.
    Then, $x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.
Then, $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x \leqslant y \Rightarrow \exists z \in \mathbb{Z}, x \leqslant z \leqslant y$.

(b) $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$

Assume $x \in \mathbb{Z}$.
    Assume $\exists y \in \mathbb{Z}, x = 3y + 1$.
        Let $y_0$ be such that $x = 3y_0 + 1$. # by assumption
        Let $y' = 3y_0^2 + 2y_0$.
        Then, $y' \in \mathbb{Z}$.
        Then, $x^2 = (3y_0+1)^2 = (3y_0)^2+2(1)(3y_0)+1^2 = 9y_0^2+6y_0+1 = 3(3y_0^2+2y_0)+1 = 3y'+1$.
        Then, $\exists y \in \mathbb{Z}, x^2 = 3y + 1$.
    Then, $(\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$.
Then, $\forall x \in \mathbb{Z}, (\exists y \in \mathbb{Z}, x = 3y + 1) \Rightarrow (\exists y \in \mathbb{Z}, x^2 = 3y + 1)$.

(c) $\neg \forall x \in \mathbb{N}, \exists y \in \mathbb{N}, y > x \wedge a_y > a_x$ — for the sequence $A = 2, 4, 6, 8, 9, 7, 5, 3, 1, 0, 0, 0, 0, 0, \ldots$

We prove the statement's negation (from the previous question).
Let $x' = 8$.
Then, $x' \in \mathbb{N}$.
Assume $y \in \mathbb{N}$.
    Assume $y > x' = 8$.
        Then, $a_y = 0$. # because $a_9 = a_{10} = a_{11} = \cdots = 0$
        Then, $a_y \leqslant 1 = a_8 = a_{x'}$.
    Then, $y > x' \Rightarrow a_y \leqslant a_{x'}$.
Then, $\forall y \in \mathbb{N}, y > x' \Rightarrow a_y \leqslant a_{x'}$.
Then, $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, y > x \Rightarrow a_y \leqslant a_x$.