Test #2 Friday, 11:10 – noon, MP103 ⟶ 8.5" × 11" aidsheet ✓
- Review: A2 (solutions posted), tutorial exercises, lecture examples if you have time left over, old test (but it has induction, you won't).
- office hours:
Wed 2-4
+ (how about ...)
Mon 3-4
Th 4-5

CSC165 winter 2013

Mathematical expression

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~heap/165/W13/
416-978-5899

Course notes, chapter 4

Outline

notes

Prove $3n^2 + 2n \in \mathcal{O}(n^2)$

Use $\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$

Then $3n^2 + 2n : \mathbb{N} \to \mathbb{R}^{\geq 0}$ # $n \in \mathbb{N} \geq 0$, so prod + sums ≥ 0

Pick $c =$ ___5___. Then $c \in \mathbb{R}^+$ # $5 \in \mathbb{R}^+$, in order to intro $\exists$

Pick $B =$ ___0___. Then $B \in \mathbb{N}$ # $0 \in \mathbb{N}$, in order to intro $\exists$

Assume n is a generic natural number # in order to intro $\forall$

Assume $n \geq B$ # in order to introduce $\Rightarrow$

Then $3n^2 + 2n \leq 3n^2 + 2n \cdot n$ # $n \cdot 2n \geq 2n$ $\forall n \in \mathbb{N}$

$= 5n^2$ # collect terms.

$= cn^2$ # $c = 5$

Then $3n^2 + 2n \leq cn^2$ # by transitivity of $\leq$

Then $n \geq B \Rightarrow 3n^2 + 2n \leq cn^2$ # introduced $\Rightarrow$

Then $\forall n \in \mathbb{N}$, # introduced $\forall$

Then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}$, # introduced $\exists$ 2x

Conclude $3n^2 + 2n \in \mathcal{O}(n^2)$ # satisfies the definition

Special case? what happens if you add a constant?

Prove $3n^2 + 2n + 5 \in \mathcal{O}(n^2)$

Use $\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$

$\vdots$

Then $3n^2 + 2n + 5 \leq 5n^2 + 5$ # prev slide

$\leq 10n^2$ # $n \geq 1$, since $B=1$

$= c\,n^2$ # $c = 10$

Look at the leading term

Prove: $7n^6 - 5n^4 + 2n^3 \in \mathcal{O}(6n^8 - 4n^5 + n^2)$ ✓ ✓

Use $\mathcal{O}(6n^8 - 4n^5 + n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c(6n^8 - 4n^5 + n^2)\}$

Then $7n^6 - 5n^4 + 2n^3 : \mathbb{N} \to \mathbb{R}^{\geq 0}$ # never negative

Pick $c = \underline{\quad 9/2 \quad}$. Then $c \in \mathbb{R}^+$

Pick $B = \underline{\quad 0 \quad}$. Then $B \in \mathbb{N}$.

assume $n \in \mathbb{N}$ # to introduce $\forall$.

assume $n \geq B$ # to intro $\Rightarrow$

Then $7n^6 - 5n^4 + 2n^3 \leq 7n^6 + 2n^3$ # omit non-pos # term

$\leq 7n^6 + 2n^3 \cdot n^3$ # $2n^3 \leq 2n^6 \; \forall n \in \mathbb{N}$

$= 9n^6$

$\leq 9n^8$ # $9n^6 \leq 9n^8 \; \forall n \in \mathbb{N}$

$= c2n^8$ # $c = 9/2$

$c(6n^8 - 4n^8) \leq c(6n^8 - 4n^5) \leq c(6n^8 - 4n^5 + n^2)$ # $n^2 \geq 0$

subtract $4n^8 \geq 4n^5, n \in \mathbb{N}$.

how to prove $n^3 \notin \mathcal{O}(3n^2)$?

Negate $\neg\left(\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow n^3 \leq c3n^2\}\right)$

$\underline{\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}}, \exists n \in \mathbb{N}, n \geq B \wedge n^3 > 3cn^2$

assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$ # in order to intro $\Rightarrow$

Pick $n = \underline{\lceil 3c \rceil + 1 + B}$. Then $n \in \mathbb{N}$.

also $n \geq B$ # at least $B+1$

Then $n^3 = n \cdot n^2 \quad > 3cn^2$

since $n > 3c$, $n \geq 3c+1$

complete bookends # + done.

Computer Science
UNIVERSITY OF TORONTO

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

If the limit evaluates to ∞, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a c, you can choose an n' with the required property.

Notes