CSC165 winter 2013

Mathematical expression

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Course notes, chapter 4

Outline

more asymptotics

notes

worst case

denote the worst-case complexity for program P with input $x \in I$, where the input size of x is n as $W_P(n) = \max\{t_P(x) \mid x \in I \wedge \text{size}(x) = n\}$

The upper bound $W_P \in \mathcal{O}(U)$ means

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B$$
$$\Rightarrow \max\{t_P(x) \mid x \in I \wedge \text{size}(x) = n\} \leq cU(n)$$

That is: $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall x \in I, \text{size}(x) \geq B$

$$\Rightarrow t_P(x) \leq cU(\text{size}(x))$$

The lower bound $W_P \in \Omega(L)$ means

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B$$
$$\Rightarrow \max\{t_P(x) \mid x \in I \wedge \text{size}(x) = n\} \geq cL(n)$$

That is: $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B$

$$\Rightarrow \exists x \in I, \text{size}(x) = n \wedge t_P(x) \geq cL(n)$$

bounding a sort

```
def IS(A) :
    """ IS(A) sorts the elements of A in non-decreasing or
    i = 1
    while i < len(A) :
        t = A[i]
        j = i
        while j > 0 and A[j-1] > t :
            A[j] = A[j-1] # shift up
            j = j-1
        A[j] = t
        i = i+1
```

n-1 times + 1 loop guard

n-1

n-1

j = i, i-1, ..., 1

n-1

n-1

I want to prove that $W_{\text{IS}} \in \mathcal{O}(n^2)$.

big-oh of n^2

We know, or have heard, that all quadratic functions grow at "roughly" the same speed. Here's how we make "roughly" explicit.

$$\mathcal{O}(n^2) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2\}$$

Those are a lot of symbols to process. They say that $\mathcal{O}(n^2)$ is a set of functions that take natural numbers as input and produce non-negative real numbers as output. An additional property of these functions is that for each of them you can find a multiplier c, and a breakpoint B, so that if you go far enough to the right (beyond B) the function is bounded above by cn^2.

In terms of limits, this says that as n approaches infinity, $f(n)$ is no bigger than cn^2 (once you find the appropriate c).

prove $W_{IS} \in \mathcal{O}(n^2)$ Prove $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, |A| = n \wedge n \geq B \Rightarrow W_{IS}(A) \leq cn^2$

Pick $c = 11$. Then $c \in \mathbb{R}^+$ # figure out later

Pick $B = 1$. Then $B \in \mathbb{N}$. # also later.

assume n is a natural number. # in order to introduce $\forall$.

Assume $|A| = n \geq B$. # in order to intro $\Rightarrow$.

Then line 1 contributes ≤ 1 step.

Then lines 2,3,4,8,9 execute $(n-1)$ times

for $i = 1, \ldots, n-1$ $= 5(n-1) + 1$ loop guard,

lines 5,6,7 contribute $3i+1$

In sum, $W_{IS}(A) \leq \underset{\text{line}}{1} + \underbrace{5(n-1)}_{2,3,4,8,9} + \underset{\text{loopguard}}{1} + \underbrace{(n-1)(3i+1)}_{5,6,7}$

$\leq 2 + 5n + n(\underline{3i+1})$.

$\leq n(5 + 3n + 1) + 2$

$= 3n^2 + 6n + 2$

$\leq 11n^2$ # since $n \geq 1$

$= cn^2$ #

prove $W_{\text{IS}} \in \Omega(n^2)$

maximum slice

```
def max_sum(L) :
  """maximum sum over slices of L"""
  max = 0
  i = 0
  while i < len(L) :
    j = i + 1
    while j <= len(L) :
      sum = 0
      k = i
      while k < j :
        sum = sum + L[k]
        k = k + 1
      if sum > max :
        max = sum
      j = j + 1
    i = i + 1
  return max
```

Notes