A1 - results on MarkUs
T1 - Friday

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Mathematical expression

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Course notes, chapter 3

Outline

non-boolean functions

notes

non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$$\lfloor x \rfloor \text{ is the largest integer} \leq x.$$

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

proving something false

n	a_n
0	0
1	0
2	1
3	1
4	2
5	2
	$\vdots$

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

$$\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$$

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

proof about limits

To prove $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$

In proving this claim you can't control the value of ε or y, but you can craft δ to make things work out.

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of δ out until you have more intuition about it.

use uniqueness

Suppose you have a predicate of the natural numbers:

$n \% 7 == 3$

$$\forall n \in \mathbb{N} \qquad S(n) \Leftrightarrow \exists k \in \mathbb{N}, n = 7k + 3$$

Is $S(3 \times 3)$ true? How do you prove that? It's useful to check out the remainder theorem from the sheet of mathematical prerequisites.

$\forall k \in \mathbb{N}, 9 \neq 7k + 3$

get wrong right

Be careful proving a claim false. Consider the claim, for some suitably defined X, Y and P, Q:

$$S: \qquad \forall x \in X, \forall y \in Y, P(x,y) \Rightarrow Q(x,y)$$

To **disprove** S, should you prove:

$$\forall x \in X, \forall y \in Y, P(x,y) \Rightarrow \neg Q(x,y)$$

What about

$$\forall x \in X, \forall y \in Y, \neg(P(x,y) \Rightarrow Q(x,y))$$

Explain why, or why not.

Define $T(n)$ by:

$$\forall n \in \mathbb{N} \qquad T(n) \Leftrightarrow \exists i \in \mathbb{N}, n = 7i + 1.$$

Take some scrap paper, **don't** write your name on it, and fill in as much of the proof of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n) \Rightarrow T(n^2)$$

Now fill in as much of the **disproof** of the following claim as possible:

$$\forall n \in \mathbb{N}, T(n^2) \Rightarrow T(n)$$

Notes

$$\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$$

Proof

Assume i is a typical natural number # in order to # introduce $\forall$

Pick $j = i+2$. Then $j \in \mathbb{N}$ # since $i, 2 \in \mathbb{N}$

Then $j = i+2 > i$ # add i to both sides of $2>0$

Also $a_j = \lfloor j/2 \rfloor = \lfloor \frac{i+2}{2} \rfloor = \lfloor i/2 + 1 \rfloor$

$> i/2$ # $\lfloor x \rfloor > x-1$

$\geq \lfloor i/2 \rfloor$ # defn of $\lfloor \ \rfloor$

$= a_i$

Then $a_j > a_i$, ie $a_j \neq a_i$

Then $j > i \wedge a_j \neq a_i$ # conjunction of above.

Then $\exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$ # exhibited $j = i+2$

Conclude $\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

Notes

To Prove: $\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$

assume $\varepsilon \in \mathbb{R}^+$

Pick $\delta =$ ___ # figure out later

assume y is a generic element of reals.

Assume $|y - \pi| < \delta$

Then $|y^2 - \pi^2| = \ldots |(y-\pi)(y+\pi)| = \underbrace{|y-\pi|}_{<\delta} \cdot |y+\pi|$

Then $|y^2 - \pi^2| < \varepsilon$

Then $|y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # introduce #$\Rightarrow$

Then $\forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$ # intro #$\forall$

Then $\exists \delta \in \mathbb{R}^+, \ldots$ # introduced $\exists$

Conclude $\forall \varepsilon \in \mathbb{R}^+, \ldots$ # introduced $\forall$