CSC165 winter 2013

Mathematical expression

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Course notes, chapter 3

Outline

non-boolean functions

notes

non-boolean functions

assume x is a generic real number # in order
to later
conclude $\forall x$

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ is the largest integer $\le x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

Do not $\forall \lfloor x \rfloor \in \mathbb{R}, \ldots$

using more of the definition

Assume

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\boxed{\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1}$$

proving something false

n	a_n
0	0
1	0
2	1
3	1
4	2
5	2

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

$\forall i \in \mathbb{N}, \exists j \in \mathbb{N}, j > i \wedge a_j \neq a_i$

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

proof about limits

Assume x is a generic real number # in order # to later # conclude $\forall x$

$\lfloor x \rfloor \leq x$ # by definition

$< x + 1$ # add x to both sides # of $0 < 1$

In proving this claim you can't control the value of ε or y, but you can craft δ to make things work out.

Then $\lfloor x \rfloor < x + 1$ # transitivity of $\leq, <$

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of δ out until you have more intuition about it.

doodle

Conclude, $\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$ # introduced $\forall$

proof about limits

Assume $x \in \mathbb{R}$ # in order to introduce $\forall$

Then $\lfloor x \rfloor \in \mathbb{Z}$ # from def.

Then $\lfloor x \rfloor + 1 \in \mathbb{Z}$ # closure, since $\lfloor x \rfloor, 1$ # $\in \mathbb{Z}$.

Well $\lfloor x \rfloor + 1 > \lfloor x \rfloor$ # add $\lfloor x \rfloor$ to $1 > 0$

Then $\lfloor x \rfloor + 1 > x$ # contrapositive! # of third clause # of definition

Or $\lfloor x \rfloor > x - 1$ # algebra

In proving this claim you can't control the value of ε or y, but you can craft δ to make things work out.

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall y \in \mathbb{R}, |y - \pi| < \delta \Rightarrow |y^2 - \pi^2| < \varepsilon$$

The claim is true. The proof format should be already familiar to you. A good approach is to fill in as much as possible, leaving the actual value of δ out until you have more intuition about it.

$$\boxed{\begin{array}{c} 0 > -1 \\ x > x - 1 \end{array}}$$

Conclude $\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$

$\lfloor x \rfloor + 1 > x$