CSC165 winter 2013

Mathematical expression

T1 – Friday – here
office hour
Today 2-4
BA 4270

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Course notes, chapter 3

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Outline

universally quantified implication, cont'd

existence

notes

proof outline

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by identation. A generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like:

Assume $x \in X$ # x is generic; what I prove applies to all of X

> Assume $P(x)$. # Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.
>
> > Then $R_1(x)$ # by previous result
> > $C2.0, \forall x \in X, P(x) \Rightarrow R_1(x)$
> > Then $R_2(x)$ # by previous result
> > $C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x)$
> >
> > $\vdots$
> >
> > Then $Q(x)$ # by previous result
> > $C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x)$
>
> Then $P(x) \Rightarrow Q(x)$ # I assumed antecedent, got consequent (aka introduced $\Rightarrow$)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ # reasoning works for all $x \in X$

a real inequality

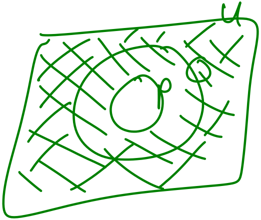
Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x + y)/2$.

some directions work better

$P \Rightarrow Q$

$\iff \neg Q \Rightarrow \neg P$

contrapositive



Prove that for any natural number n, n^2 odd implies that n is odd.

proving existence

To prove the a set is non-empty, it's enough to exhibit one element. How do you prove:

Sometimes x, No

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$

Proof

Pick $x = 2$.

Then $x \in \mathbb{R}$ # 2 well-known $\in \mathbb{R}$.

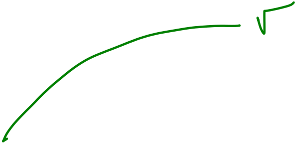
Then $x^3 + 3x^2 - 4x = 8 + 12 - 8$ # since $x = 2$

$= 12$

Then $\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$ # e.g. $x = 2$

introduce $\exists$

prove a claim about a sequence



n	a_n
0	0
1	1
2	4
3	9
4	16
⋮	⋮

Define sequence a_n by:

$$\forall n \in \mathbb{N} \qquad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$

Proof: Pick $i = 4$. Then $i \in \mathbb{N}$ # 'cause 4 is.

Assume $j \in \mathbb{N}$ # generic, in order to introduce $\forall$.

Assume $a_j = j^2 \leq i = 4$ # antecedent.

Then $\sqrt{j^2} = j \leq \sqrt{i} = 2$ # $\sqrt{\ }$ strictly monotonic.

$< 4 = j$ # $2 < 4$

Then $j < i$

tie up all those loose ends

contradiction — a special case of contrapositive

$F_1 \wedge F_2 \wedge \cdots \wedge F_{3978238} \Rightarrow S$

$\neg S \Rightarrow \neg F_1 \vee \neg F_2 \vee \cdots \vee \neg F_{\ldots}$

Define the prime natural numbers as
$P = \{p \in \mathbb{N} \mid p$ has exactly two distinct divisors in $\mathbb{N}\}$. How do you prove:

$$S: \qquad \forall n \in \mathbb{N}, |P| > n$$

It would be nice to have some result R that leads to S. If you could show $R \Rightarrow S$, and that R is true, then you'd be done. But, out of many elementary results, how do you choose an R? Contradiction will often lead you there.

Assume $\neg S$, $\exists n \in \mathbb{N}, |P| \leq n$

non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ *is the largest integer* $\leq x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

Notes

Assume $\neg S$, $\exists n \in \mathbb{N}, |P| \leq n$

Then $\exists k \in \mathbb{N}, |P| = k$ # $0 \leq k \leq n$.

Then $P = \{p_1, p_2, \ldots, p_k\}$

Then $m = p_1 \cdot p_2 \cdots p_k \in \mathbb{N}$ # $\mathbb{N}$ closed # under X

Then $m \geq 6$ # $2, 3 \in P$

> 1 # and $p_i \geq 1$

Then $m+1 > 1$ # since $m > 1$

Then $\exists p \in P, p \mid m+1$ # every natural # >1 has prime # factor,

also $p \mid m$ # since m is product of all # primes in P.

Then $p \mid ((m+1) - m)$, or $p \mid 1$

$\Rightarrow p = 1$ — contradiction!

1 is not prime

Then, since assume $\neg S$ leads to contradiction, S must be true

m, not 1 as wrote in lecture!