CSC165 winter 2013

Mathematical expression

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Course notes, chapter 3

Outline

universally quantified implication, cont'd

existence

notes

proof outline

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by identation. A generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like:

Assume $x \in X$ # x is generic; what I prove applies to all of X

  Assume $P(x)$. # Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.

    Then $R_1(x)$ # by previous result $C2.0, \forall x \in X, P(x) \Rightarrow R_1(x)$
    Then $R_2(x)$ # by previous result $C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x)$

    $\vdots$

    Then $Q(x)$ # by previous result $C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x)$

  Then $P(x) \Rightarrow Q(x)$ # I assumed antecedent, got consequent (aka introduced $\Rightarrow$)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ # reasoning works for all $x \in X$

a real inequality

Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x + y)/2$.

some directions work better

Prove that for any natural number n, n^2 odd implies that n is odd.

proving existence

To prove the a set is non-empty, it's enough to exhibit one element. How do you prove:

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$

prove a claim about a sequence

Define sequence a_n by:

$$\forall n \in \mathbb{N} \qquad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$

contradiction — a special case of contrapositive

Define the prime natural numbers as
$P = \{p \in \mathbb{N} \mid p$ has exactly two distinct divisors in $\mathbb{N}\}$. How do you prove:

$$S : \qquad \forall n \in \mathbb{N}, |P| > n$$

It would be nice to have some result R that leads to S. If you could show $R \Rightarrow S$, and that R is true, then you'd be done. But, out of many elementary results, how do you choose an R? Contradiction will often lead you there.

non-boolean functions

Take care when expressing a proof about a function that returns a non-boolean value, such as a number:

$\lfloor x \rfloor$ *is the largest integer* $\leq x$.

Now prove the following statement (notice that we quantify over $x \in \mathbb{R}$, not $\lfloor x \rfloor \in \mathbb{R}$:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor < x + 1$$

using more of the definition

You may have been disappointed that the last proof used only part of the definition of floor. Here's a symbolic re-writing of the definition of floor:

$$\forall x \in \mathbb{R} \qquad y = \lfloor x \rfloor \Leftrightarrow y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}, z \leq x \Rightarrow z \leq y)$$

The full version of the definition should prove useful to prove:

$$\forall x \in \mathbb{R}, \lfloor x \rfloor > x - 1$$

proving something false

Define a sequence:

$$\forall n \in \mathbb{N} \qquad a_n = \lfloor n/2 \rfloor$$

(of course, if you treat "/" as integer division, there's no need to take the floor. Now consider the claim:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i \Rightarrow a_j = a_i$$

The claim is false. Disprove it.

proof by cases

Sometimes your argument has to split to take into account possible properties of your generic element:

$$\forall n \in \mathbb{N}, n^2 + n \text{ is even}$$

A natural approach is to factor $n^2 + n$ as $n(n+1)$, and then consider the case where n is odd, then the case where n is even.

Notes