CSC165 winter 2013

Mathematical expression

aid sheet 8.5" x 11", hand written, both sides, all 4 edges.

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)

http://www.cdf.toronto.edu/~heap/165/W13/

416-978-5899

Ch 1+2
- quantifiers
- implications
- negations
- English ⟺ symbolic

Course notes, chapter 3

NB Please know list comprehensions
[x*x for x in L]

Outline

universally quantified implication, cont'd

existence

notes

proof outline

$\forall x, y \in \mathbb{R}^{\geq 0}, x > y \Rightarrow (x+y)/2 > \sqrt{xy}$

More flexible format required in this course. Each link in the chain justified by mentioning supporting evidence in a comment beside it. Here are portions of an argument where scope of assumption is shown by identation. A generic proof that $\forall x \in X, P(x) \Rightarrow Q(x)$ might look like:

Assume $x \in X$ # x is generic; what I prove applies to all of X

Assume $P(x)$. # Antecedent. Otherwise, $\neg P(x)$ means we get the implication for free.

Then $R_1(x)$ # by previous result
$C2.0, \forall x \in X, P(x) \Rightarrow R_1(x)$
Then $R_2(x)$ # by previous result
$C2.1, \forall x \in X, R_1(x) \Rightarrow R_2(x)$

$\vdots$

Then $Q(x)$ # by previous result (much) later.
$C2.n, \forall x \in X, R_n(x) \Rightarrow Q(x)$

Then $P(x) \Rightarrow Q(x)$ # I assumed antecedent, got consequent (aka introduced $\Rightarrow$)

Then $\forall x \in X, P(x) \Rightarrow Q(x)$ # reasoning works for all $x \in X$

a real inequality

$$\forall x, y \in \mathbb{R}^{\geq 0}, \ x > y \Rightarrow (x+y)/2$$

Prove that for every pair of non-negative real numbers (x, y), if x is greather than y, then the geometric mean, $\sqrt{xy}$ is less than the arithmetic mean, $(x+y)/2$.

some directions work better

already proved
n odd $\Rightarrow n^2$ odd

$\forall n \in \mathbb{N}, n^2 \text{ odd} \Rightarrow n \text{ odd}$

Prove that for any natural number n, n^2 odd implies that n is odd.

proving existence

To prove the a set is non-empty, it's enough to exhibit one element. How do you prove:

$$\exists x \in \mathbb{R}, x^3 + 3x^2 - 4x = 12$$

prove a claim about a sequence

Define sequence a_n by:

$$\forall n \in \mathbb{N} \qquad a_n = n^2$$

Now prove:

$$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, a_j \leq i \Rightarrow j < i$$

contradiction — a special case of contrapositive

Define the prime natural numbers as $P = \{p \in \mathbb{N} \mid p \text{ has exactly two distinct divisors in } \mathbb{N}\}$. How do you prove:

$$S : \qquad \forall n \in \mathbb{N}, |P| > n$$

It would be nice to have some result R that leads to S. If you could show $R \Rightarrow S$, and that R is true, then you'd be done. But, out of many elementary results, how do you choose an R? Contradiction will often lead you there.

Notes

$\forall x, y \in \mathbb{R}^{\geq 0}, x > y \Rightarrow (x+y)/2 > \sqrt{xy}$

assume x, y are generic, non-negative real numbers.

Assume $x > y$

Then $x - y > 0$ # subtract y both sides

Then $x^2 + y^2 - 2xy > 0$ # mult by +-ve.

Then $x^2 + y^2 + 2xy > 4xy$ # add $4xy$ to both.

$(x+y)^2 > 4xy$ # algebra.

$x + y > 2\sqrt{xy}$ # $\sqrt{\ }$ is monotonic.

$\frac{x+y}{2} > \sqrt{xy}$ # divide by 2.

$\frac{x+y}{2} > \sqrt{xy}$ # by previous line

Then $x > y \Rightarrow \frac{x+y}{2} > \sqrt{xy}$ # introduced $\Rightarrow$

Then, for all pairs of non-negative real numbers x, y, $x > y \Rightarrow \frac{x+y}{2} > \sqrt{xy}$ # introduced $\forall$

Notes

$\forall n \in \mathbb{N},\ n^2 \text{ odd} \Rightarrow n \text{ odd}$

Assume n is a generic element of $\mathbb{N}$

assume n^2 odd # antecedent

Then $\exists k \in \mathbb{N},\ n^2 = 2k+1$ # definition n^2 odd

$k = 2(k')^2 + k'$ hard to take square root.

- ok try. $n^2 - 1 = 2k$

So $(n-1)(n+1) = 2k$ (even).

need heavy machinery if prime (2) divides mn, then $2|m$ or $2|n$

Then $\exists k' \in \mathbb{N},\ n = 2k'+1$, ie n odd.

Then n^2 odd $\Rightarrow$ n odd # introduced $\Rightarrow$

Conclude that whenever $n \in \mathbb{N}$, n^2 odd $\Rightarrow$ n odd.

introduce $\forall$

Notes

Assume $n \in \mathbb{N}$ # generic.

Assume n not odd (ie even) # recall # $P \Rightarrow Q$ # negation # of consequent

Then $\exists k \in \mathbb{N}, n = 2k$ # by definition

Then $n^2 = 4k^2$ # squaring

$= 2(2k^2)$ # algebra

Then $\exists k' \in \mathbb{N}, n^2 = 2k'$ # $k' \doteq 2k^2 \in \mathbb{N}$

since $2, k \in \mathbb{N}$

$\mathbb{N}$ closed under $\times$

Then n^2 is even

Then n^2 is not odd # negation of # antecedent

Then n^2 odd $\Rightarrow$ n odd (introduced $\Rightarrow$ by proving contrapositive)!

Conclude, $\forall n \in \mathbb{N}, n^2$ odd $\Rightarrow$ n odd