CSC165 winter 2013

Mathematical expression

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Course notes, chapter 2–3

Outline

implication as disjunction

mixed quantifiers

Notes

implication two ways

The result of the following truth table is useful enough to bear restating:

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

bi-implication

Translate bi-implication into the conjunction of two disjunctions:

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$(\neg P \vee Q) \wedge (\neg Q \vee P)$

Now change your expression for bi-implication into the disjunction of two conjunctions (use the some of the equivalences from a few slides ago)

$\equiv ((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P) \equiv \neg P \wedge \neg Q \vee Q \wedge \neg Q \vee \neg P \wedge P \vee Q \wedge P$

$\equiv \neg P \wedge \neg Q \vee Q \wedge P$

What's the negation of bi-implication? How would you explain it in English?

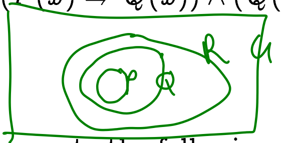
$P \wedge \neg Q \vee Q \wedge \neg P$ — exclusive OR!

$\rightarrow \oplus$ XOR

transitivity

What does the following statement mean, when you interpret it as a venn diagram?

$$\forall x \in X, (P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$$



ie $P(x) \Rightarrow R(x)$

For another insight, negate the following statement, and simplify it by transforming implications into disjunctions:

$$\neg \left[((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R) \right]$$

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \wedge P \wedge \neg R$$

$$\equiv \left[(\neg P \vee Q) \wedge (\neg Q \vee R) \right] \wedge P \wedge \neg R$$

for all, one. . . one for all

What's the difference between these two claims:

$$\forall x \in L1, \exists y \in L2, x + y = 5$$

$$\exists y \in L2, \forall x \in L1, x + y = 5$$

```
def P(x,y) : return x + y == 5
L1 = L2 = [1, 2, 3, 4]

def forallExists(P, L1, L2) :
    return False not in [True in [P(x,y) for y in L2] for x in L1]

def existsForall(P, L1, L2) :
    return True in [False not in [P(x,y) for x in L2] for y in L1]
```

Can you switch $\forall \varepsilon \in \mathbb{R}^+$ with $\exists \delta \in \mathbb{R}^+$ without altering the truthfulness of the statement below?

$$\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon$$

$\forall x \in \mathbb{R}, \exists \delta \ldots, \forall \varepsilon \_, |x-0.6| < \delta \Rightarrow |\quad| < \varepsilon$

(you can!). How about: $\delta = |x - 0.6|/2$) cheap! vacuous truth!

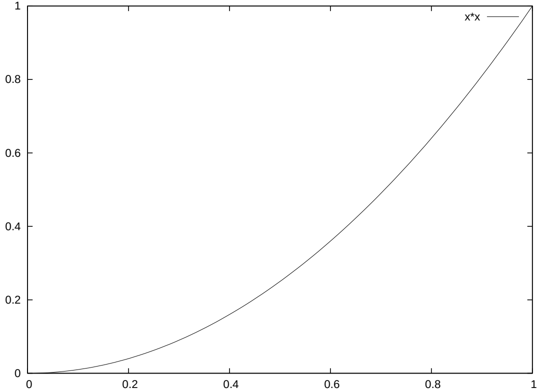
$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 0.6| < \delta \Rightarrow |x^2 - 0.36| < \varepsilon$$

This latter is often written in a different form:

$$\lim_{x \to 0.6} x^2 = 0.36$$

First specify how close to 0.36 x^2 has to be (ε), then I can choose how close to 0.6 x must be (δ). If I choose δ first, can it work for all ε?

graphically...



are we close to infinity yet?

What is meant by phrases such as "as x approaches (gets close to) infinity, x^2 increases without bound (sometimes 'becomes infinite')"? Or even

$$\lim_{x \to \infty} x^2 = \infty$$

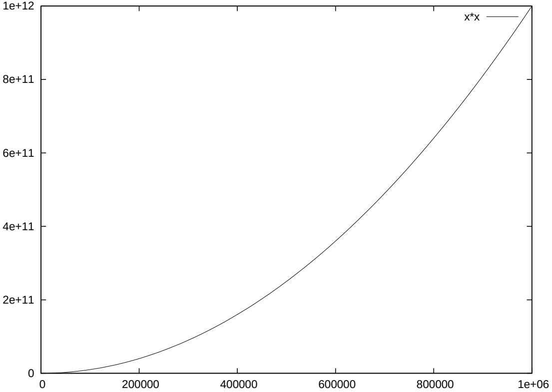
Look at the graph of x^2. Do either x or x^2 ever reach infinity?

How about:

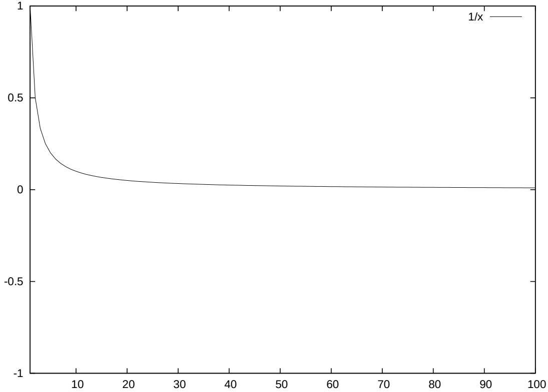
$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}, x > \delta \Rightarrow x^2 > \varepsilon$$

Getting "close" to infinity means getting far from (and greater than) zero. Once you have a specification for how far from zero x^2 must be (ε), you can come up with how far from zero x must be (δ). Can you choose a δ in advance that works for all ε?

graph “approaching infinity”



asymptotic



double quantifiers

There are (at least) three ways to claim that a certain subset of the cartesian product $\mathbb{N} \times \mathbb{N}$, **aka** $\mathbb{N}^2$ is non-empty:

$$\exists m \in \mathbb{N}, \exists n \in \mathbb{N}, m^2 = n$$
$$\exists (m, n) \in \mathbb{N}^2, m^2 = n$$
$$\exists n \in \mathbb{N}, \exists m \in \mathbb{N}, m^2 = n$$

Whether we think of this as a statement about a subset of the cartesian product being empty, or a relation between non-empty subsets of $\mathbb{N}$, it is symmetrical.
There are (at least) three ways to claim that the entire cartesian product $\mathbb{N} \times \mathbb{N}$ has some property:

$$\forall m \in \mathbb{N}, \forall n \in \mathbb{N}, mn \in \mathbb{N}$$
$$\forall (m, n) \in \mathbb{N}^2, mn \in \mathbb{N}$$
$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, mn \in \mathbb{N}$$

Again, the order in which we consider elements of an ordered pair doesn't change the logic.

Notes

$$\neg\left[\left((P \Rightarrow Q) \wedge (Q \Rightarrow R)\right) \wedge P \wedge \neg R\right]$$

$$\equiv \left[(\neg P \vee Q) \wedge (\neg Q \vee R)\right] \wedge P \wedge \neg R$$

$$\equiv \left[\left((\neg P \vee Q) \wedge \neg Q\right) \vee (\neg P \vee Q) \wedge R\right] \wedge P \wedge \neg R$$

$$\equiv \left[\neg P \wedge \neg Q \vee \underbrace{Q \wedge \neg Q}_{F} \vee \neg P \wedge R \vee Q \wedge R\right] \wedge (P \wedge \neg R)$$

$$\equiv \underbrace{\neg P \wedge \neg Q \wedge P \wedge \neg R}_{F} \vee \underbrace{\neg P \wedge R \wedge P \wedge \neg R}_{F} \vee \underbrace{Q \wedge R \wedge P \wedge \neg R}_{F}$$

$$\equiv F$$

Notes

Freebie

$P \Rightarrow (Q \Rightarrow R) \equiv Q \Rightarrow (P \Rightarrow R)$

Prove

you may use truth tables or De ~~Morgan's~~ + other identities ...

$\neg P \vee (\neg Q \vee R)$ # $\Rightarrow$ identity

$\equiv (\neg P \vee \neg Q) \vee R$ # associative

$\equiv (\neg Q \vee \neg P) \vee R$ # commutative

$\equiv \neg Q \vee (\neg P \vee R)$ # associative

$\equiv Q \Rightarrow (P \Rightarrow R)$ # $\Rightarrow / \vee$ identity

done