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Mathematical expression

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Course notes, chapter 2

Outline

leftovers. . .

conjunction, disjunction

negation

Notes

“natural” language

Translate “unless” as “if not”:

Don’t knock it unless you’ve tried it.

idiom

Some expressions for restricting domains are more common than others.

- "Every D that is a P is also a Q." Usually $\forall x \in D, P(x) \Rightarrow Q(x)$. Less common $\forall x \in D \cap P, Q(x)$. What about $\forall x \in D, P(x) \wedge Q(x)$ ($\wedge$ means "and")?

- "Some D that is a P is also a Q." Usually $\exists x \in D, P(x) \wedge Q(x)$. Less common $\exists x \in D \cap P \cap Q$. What about $\exists x \in D, P(x) \Rightarrow Q(x)$?

conjunction: $\wedge$

Combine two statements by claiming they are both true with logical "and":

$A(x)$ and $B(x)$ (python keyword **and** works like this)

$A(x) \wedge B(x)$ ($\wedge$ is a symbol for "and")

As sets: $x \in A \cap B$

Notice that a conjunction is **false** if either part is false. "The employee makes less than 100,000 and more than 60,000," is true for Gwen, but false for Ellen.

Employee	Gender	Salary
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

watch out for English "and"

Sometimes the English word "and" is used to smear some meaning over several components:

There is a pen and a telephone.

In the universe of objects, O, with predicates $P(x)$ (x is a pen) and $T(x)$ (x is a telephone), you could try to translate this as $\exists x \in O, P(x) \wedge T(x)$. What's a better translation into symbols?

Occasionally English usage of **and** will differ from logical usage even in mathematical material:

The solutions are $x < 10$ *and* $x > 20$
The solutions are $x < 20$ *and* $x > 10$

The first statements probably meant the union of the two sets, or the logical **or**. The second meant the intersection, so the logical **and** is appropriate.

disjunction: $\vee$

Combine two statements by claiming that at least one of them is true using **or** ($\vee$ in symbols).

$A(x)$ or $B(x)$ (the python keyword or works like this)

$A(x) \vee B(x)$ (in symbols)

$x \in A \cup B$ (as sets)

Notice the close connection between the symbols for conjunction and intersection, $\wedge$, $\cap$, and the symbols for disjunction and union, $\vee$, $\cup$. Coincidence? In any case, you may use it as a mnemonic.

"The employee is female or earns more than 35,000."

Employee	**Gender**	**Salary**
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
[illegible]	[illegible]	[illegible]

silly English tricks

In logic we use **or** generously, or inclusively, to mean something like "and/or". Sometimes we convey the **inclusive or** by saying something like "A or B, or both." Be aware that natural English sometimes uses or to mean "A or B, but not both" — something we'd call **exclusive or** in logic:

> *Either we play the game my way, or I'm taking my ball and going home.*

negation: $\neg$

Negate the statement "All employees earning over 110,000 are female." Usually prepending the word "Not" will work, and in logic we use the corresponding symbol $\neg$:

$$\neg(\forall e \in E, O(e) \Rightarrow F(e))$$

A good exercise is to "work" the negation $\neg$ as far into the statement as possible. The statement is true exactly when its negation is false.

The original statement is universally quantified, so it says something about an absence of counterexamples. The negation of the original statement should claim something about the presence of counterexamples.

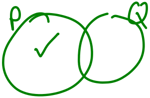
special negation idiom

$\neg(P(x) \Rightarrow Q(x))$

Negating implications is a common task. There are several equivalent ways of doing this, but some are more common than others. Try negating the following in such a way that the $\neg$ symbol applies to the "smallest possible" part of the expression:

negate → $\neg \left(\forall x \in X, P(x) \Rightarrow Q(x) \right)$

$\exists x \in X, P(x) \wedge \neg Q(x)$



Now for symmetry, negate the following in such a way that the $\neg$ symbol applies to the "smallest possible" part of the expression:

$$\neg \left(\exists x \in X, P(x) \wedge \neg Q(x) \right)$$

$\forall x \in X, P(x) \Rightarrow Q(x)$.

standard negation

Negated expressions have some standard transformations:

- $\neg \forall x \in X, \ldots \Leftrightarrow \exists x \in X, \neg \ldots$
- $\neg \exists x \in X, \ldots \Leftrightarrow \forall x \in X, \neg \ldots$
- $\neg(P(x) \Rightarrow Q(x)) \Leftrightarrow P(x) \wedge \neg Q(x)$
- $\neg(P(x) \wedge Q(x)) \Leftrightarrow P(x) \Rightarrow \neg Q(x)$ (has this become asymmetrical?)

Push the $\neg$ symbol "as far in" to the following expression as possible:

$$\neg(\forall x \in X, \exists y \in Y, P(x) \Rightarrow Q(y))$$

$$\exists x \in X, \forall y \in Y, P(x) \wedge \neg Q(y)$$

scope

In order to parse a logical expression we need to know which subexpressions to parse first. Although there's often conventions (just as in your favourite programming language), such as evaluating $\wedge$ before $\Rightarrow$, when in doubt you should use parentheses:

$$(P(x) \vee Q(x)) \Rightarrow R(x) \quad \text{versus} \quad P(x) \vee (Q(x) \Rightarrow R(x))$$

This becomes particularly important when you want to be explicit about the scope of universal quantifiers:

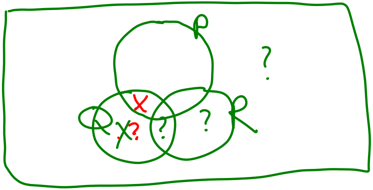
$$(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y) \Rightarrow (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 < y)$$

Notice that the scope of the quantification is inside the relevant parentheses. There's no reason that the y in the antecedent would be the same as the y in the consequent. It could be re-written:

$$(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x < y) \Rightarrow (\forall z \in \mathbb{R}, \exists w \in \mathbb{R}, z^2 < w)$$

drawing truth

As conjunctions, disjunctions, negations, and other combinations of predicates become more ornate, we need help to interpret them. To think about the truth value of up to three predicates, you can probably draw a venn diagramm. For example, draw the venn diagram showing which region(s) must **not** have any elements to still remain consistent with $P(x) \Rightarrow (Q(x) \Rightarrow R(x))$.



How would you draw the analogous diagram for predicates P, Q, R, and S? Perhaps if your 3D rendering skills were pretty good you'd manage. However, to combine more predicates, you need a new tool.

tabulating truth

The standard venn diagram for 3 sets has 2^3 regions — one region for each possible combination of truth values for its component sets. We can get the same effect with a rectangular diagram, or table:

all possibilities (at least for truth P, Q, R.

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

As an exercise, compare this to the table for $(P \wedge Q) \Rightarrow R$. What do you conclude?

tautology, satisfiability, unsatisfiability

You may have been unsettled in the previous slides that there were no domains stated for P, Q, or R, no definitions for them, and nothing about what arguments (if any) these predicates take. The reason this was okay was that we considered all 8 possible truth values for P, Q, and R — all possible logical "worlds" that matter in their case.
An example to help think about this is to consider all possible domains D that P or Q could be part of, and all possible meanings for predicates P or Q. Consider this **very** general situation:

$$\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall Q \in \mathcal{P}(D), \forall x \in D,$$
$$(P(x) \implies Q(x)) \iff (\neg P(x) \vee Q(x))$$

Although there are infinitely many domains in $\mathcal{D}$, and infinitely many meanings for predicates P and Q, there are only four lines in the relevant truth table, and the statement is true in all four.

weaker, weirder

The situation on the previous slide was a tautology — the statement is true in every possible world.

$$\exists D \in \mathcal{D}, \exists P \in \mathcal{P}(D), \exists Q \in \mathcal{P}(D), \exists x \in D,$$
$$(P(x) \implies Q(x)) \iff (Q(x) \implies P(x))$$

Sometimes

. . . it's possible to concoct a world where the statement is true. We say it's satisfiable.

What about a statement that can't every be true no matter what world we devise:

$$\forall D \in \mathcal{D}, \forall P \in \mathcal{P}(D), \forall x \in D, (P(x) \wedge \neg P(x))$$

never true!

. . . This is unsatisfiable (aka a contradiction).

commutative, associative, distributive

Some laws of arithmetic have counterparts in logic and set operations:

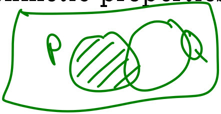
$$P \wedge Q \Leftrightarrow Q \wedge P \qquad P \vee Q \Leftrightarrow Q \vee P$$

$$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R \qquad P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$$

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R) \qquad P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

$$P \wedge (Q \vee \neg Q) \Leftrightarrow P \Leftrightarrow P \vee (Q \wedge \neg Q) \qquad P \wedge P \Leftrightarrow P \Leftrightarrow P \vee P$$

Convince yourself that the identities above are true using venn diagrams, truth tables, or expressing them in words. Some are analogous to arithmetic properties of numbers. Some are truly novel.



De Morgan's Law(s)

Really just one law, but you switch the rôles of $\wedge$ and $\vee$:

$$\begin{aligned} \neg(P \vee Q) &\Leftrightarrow \neg P \wedge \neg Q \\ \neg(P \wedge Q) &\Leftrightarrow \neg P \vee \neg Q \end{aligned}$$

Again, you should draw venn diagrams and fill in truth tables to convince yourself this is true. Using associativity and commutativity, you can extend these laws to conjunctions and disjunctions of more than two expressions.

Notes