T2 — done well, back @ ≈ 11:45
A2 — B⁻ — some grading not yet done
A3 — up tonight

CSC165 winter 2013

Mathematical expression

Danny Heap
heap@cs.toronto.edu
BA4270 (behind elevators)
http://www.cdf.toronto.edu/~heap/165/W13/
416-978-5899

Course notes, chapter 4

Outline

Bounded below

some theorems

notes

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

If the limit evaluates to ∞, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a c, you can choose an n' with the required property.

prove $2^n \notin \mathcal{O}(n^2)$

Using l'Hôpital's rule and limits

big-Omega

Notice that the definition of big-Omega differs in just one character from big-Oh:

$$\Omega(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq cg(n)\}$$

The rôle of B is, as with big-Oh, to act as a breakpoint, so comparisons don't have to start at the origin.

The rôle of c is to scale g *down* below f.

If you're proving $f \in \Omega(g)$, you get to choose c and B to suit your proof. Notice that it would be really unfair to allow c to be zero.

one more bound

It often happens that functions are bounded above *and* below by the same function. In other words, $f \in \mathcal{O}(G) \wedge f \in \Omega(g)$. We combine these two concepts into $f \in \Theta(g)$.

$$\Theta(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N},$$
$$\forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

You might want to draw pictures, and conjecture about appropriate values of c_1, c_2, B for $f = 5n^2 + 15$ and $g = n^2$.

$\forall f, g, h \in \mathcal{F}$. $\mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}^{\geq}\}$

How to prove general statement about two functions:

$\forall f, g, h \in \mathcal{F}, (f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

assume $f, g, h \in \mathcal{F}$, $f \in \mathcal{O}(g)$, $g \in \mathcal{O}(h)$ # introduce $\forall$ and # introduce $\Rightarrow$

$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c g(n)$ # $f \in \mathcal{O}(g)$

$\exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B' \Rightarrow g(n) \leq c' h(n)$ # $g \in \mathcal{O}(h)$.

Pick $c'' = c \cdot c'$. Then $c'' \in \mathbb{R}^+$

Pick $B'' = B + B'$. Then $B'' \in \mathbb{N}$.

assume $n \in \mathbb{N}$ and $n \geq B''$ # intro $\forall, \Rightarrow$ again.

Then $f(n) \leq c g(n)$ # $f \in \mathcal{O}(g)$ # $n \geq B$

$\leq c c' h(n)$ # $g \in \mathcal{O}(h)$ mult $g(n) \leq c' h(n)$ # by c # $n \geq B'$

$= c'' h(n)$ # $c'' = c \cdot c'$

Conclude $\forall f, g, h \in \mathcal{F}, (f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

showed $\forall, \Rightarrow, \exists, \forall, \Rightarrow$ in appropriate spots.

$\forall f, g \in \mathcal{F}$ $\qquad \mathcal{F} = \{f : \mathbb{N} \to \mathbb{R}^{\geq 0}\}$

how about:

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow g \in \Omega(f)$

assume $f, g \in \mathcal{F}$, $f \in \mathcal{O}(g)$. # into $\forall$ and $\Rightarrow$

then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq c\,g(n)$ # $f \in \mathcal{O}(g)$.

Must show $\exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B' \Rightarrow g(n) \geq c' f(n)$

Pick $c' = 1/c$. Then $c' \in \mathbb{R}^+$

Pick $B' = B$. Then $B' \in \mathbb{N}$.

assume $n \in \mathbb{N}$ and $n \geq B'$ # into $\forall$, into $\Rightarrow$

Then $c\,g(n) \geq f(n)$ # since $n \geq B$, $f \in \mathcal{O}(g)$

Then $g(n) \geq \frac{1}{c} f(n)$ # mult both sides $\frac{1}{c} \in \mathbb{R}^+$.

$= c' f(n)$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow g \in \Omega(f)$

prove or disprove:

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \cdot f \in \mathcal{O}(g \cdot g)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, (f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h)) \Rightarrow (f + g) \in \mathcal{O}(h)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \in \mathcal{O}(g \cdot g)$

Notes