CSC165 winter 2013

Mathematical expression

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Course notes, chapter 4
(big-Oh, big-Ω, big-Θ.)

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Outline

Bounded below

some theorems

notes

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

2^n n^2

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$$

If the limit evaluates to ∞, then that's the same as saying:

$$\underbrace{\forall c \in \mathbb{R}^+}_{\text{no matter how big}}, \underbrace{\exists n' \in \mathbb{N}}_{\text{go far enough right}}, \underbrace{\forall n \in \mathbb{N}}, n \geq n' \Rightarrow \underbrace{\frac{2^n}{n^2} > c}$$

Once your enemy hands you a c, you can choose an n' with the required property.

prove $2^n \notin \mathcal{O}(n^2)$

negate $2^n \in \mathcal{O}(n^2)$

$\neg(\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2^n \leq cn^2)$

Using l'Hôpital's rule and limits

$\forall c \in \mathbb{R}^+, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \wedge 2^n > cn^2$

assume $c \in \mathbb{R}^+$, assume $B \in \mathbb{N}$ # to intro $\forall, \forall$.

Then $\lim_{n\to\infty} \frac{2^n}{n^2} = \infty$ # by l'Hopital's rule, = # $\lim_{n\to\infty} \frac{(\ln 2 \cdot 2^n)'}{(2 \cdot n)'} = \lim_{n\to\infty} \frac{\ln 2 \cdot \ln 2 \cdot 2^n}{2}$ # ∞

So, $\exists n' \in \mathbb{N}, \forall n \geq n', \frac{2^n}{n^2} > c$ # by meaning of lim

Pick $n = B + n'$. Then $n \in \mathbb{N}$; $B, n' \in \mathbb{N}$.

Then $n \geq B$.

Also $2^n/n^2 > c$ # By choice of n

So $2^n > cn^2$.

big-Omega

Prove $f(n) = n^2$, $g(n) = 15n^2 + 5$, then $f \in \Omega(g)$

Notice that the definition of big-Omega differs in just one character from big-Oh:

only

$$\Omega(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \boxed{\geq} cg(n)\}$$

The rôle of B is, as with big-Oh, to act as a breakpoint, so comparisons don't have to start at the origin.

$\cancel{15n^2 + 5 \in O(n^2)}$

choose $0.06 = c$

The rôle of c is to scale g *down* below f.

show $n^2 \in \Omega(15n^2 + 5)$

choose $c = 0.06$

If you're proving $f \in \Omega(g)$, you get to choose c and B to suit your proof.

Notice that it would be really unfair to allow c to be zero.

one more bound

$f(n) = 15n^2 + 5$
$g(n) = n^2$

It often happens that functions are bounded above *and* below by the same function. In other words, $f \in \mathcal{O}(G) \wedge f \in \Omega(g)$. We combine these two concepts into $f \in \Theta(g)$.

Pick $c_1 =$ ______
Pick $c_2 =$ ______
Pick $B =$ ______

two multipliers!

$$\Theta(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

assume $n \in \mathbb{N}$

$$0 < c_1 \leq \frac{f(n)}{g(n)} \leq c_2$$

You might want to draw pictures, and conjecture about appropriate values of c_1, c_2, B for $f = 5n^2 + 15$ and $g = n^2$.

How to prove general statement about two functions:

$(f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

how about:

$f \in \mathcal{O}(g) \Rightarrow g \in \Omega(f)$

prove or disprove:

$f \in \mathcal{O}(g) \Rightarrow f \cdot f \in \mathcal{O}(g \cdot g)$

prove or disprove:

$(f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h)) \Rightarrow (f + g) \in \mathcal{O}(h)$

prove or disprove:

$f \in \mathcal{O}(g) \Rightarrow f \in \mathcal{O}(g \cdot g)$

Notes

Pick $c_1 =$ 15 Then $c_1 \in \mathbb{R}^+$

Pick $c_2 =$ 20 Then $c_2 \in \mathbb{R}^+$

Pick $B =$ 1 Then $B \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ # to introduce $\forall$.

Assume $n \geq B$. # to intro $\forall$.

Then $f(n) = 15n^2 + 5$

$\leq 15n^2 + 5n^2 = 20n^2$ # $n \geq 1$

$= c_2 n^2$ # $c_2 = 20$

Also $f(n) = 15n^2 + 5$

$\geq 15n^2$ # omit +ve term

$= c_1 n^2$ # $c_1 = 15$

Finish up bookends