CSC165 winter 2013

Mathematical expression

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writes this weekend -

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Course notes, chapter 4

Outline

Bounded below

some theorems

notes

non-polynomials

Big-oh statements about polynomials are pretty easy to prove: $f \in \mathcal{O}(g)$ exactly when the highest-degree term of g is no smaller than the highest-degree term of f.

What about functions such as $\log(n)$ or 3^n? Logarithmic functions are in big-Oh of **any** polynomial, whereas exponential functions (with a base bigger than 1) are not in big-Oh of any polynomial. How do you prove such things?

Prove $2^n \notin \mathcal{O}(n^2)$

Use $\lim_{n\to\infty} 2^n/n^2$

Do you know anything about the ratio $2^n/n^2$, as n gets very large? How do you evaluate:

$$\lim_{n\to\infty} \frac{2^n}{n^2}$$

If the limit evaluates to ∞, then that's the same as saying:

$$\forall c \in \mathbb{R}^+, \exists n' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n' \Rightarrow \frac{2^n}{n^2} > c$$

Once your enemy hands you a c, you can choose an n' with the required property.

prove $2^n \notin \mathcal{O}(n^2)$

Using l'Hôpital's rule and limits

big-Omega

Notice that the definition of big-Omega differs in just one character from big-Oh:

$$\Omega(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c \in \mathbb{R}^{+}, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \geq cg(n)\}$$

The rôle of B is, as with big-Oh, to act as a breakpoint, so comparisons don't have to start at the origin.

The rôle of c is to scale g *down* below f.

If you're proving $f \in \Omega(g)$, you get to choose c and B to suit your proof.

Notice that it would be really unfair to allow c to be zero.

one more bound

It often happens that functions are bounded above *and* below by the same function. In other words, $f \in \mathcal{O}(G) \wedge f \in \Omega(g)$. We combine these two concepts into $f \in \Theta(g)$.

$$\Theta(g) = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0} \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \\ \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

You might want to draw pictures, and conjecture about appropriate values of c_1, c_2, B for $f = 5n^2 + 15$ and $g = n^2$.

How to prove general statement about two functions:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g, h \in \mathcal{F}, (f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h)) \Rightarrow f \in \mathcal{O}(h)$

how about:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow g \in \Omega(f)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \cdot f \in \mathcal{O}(g \cdot g)$

$f(n) \leq c\, g(n)$

$f(n) \cdot f(n) \overset{?}{\leq} c' g(n) \cdot g(n)$

$f(n) \leq c\, g(n)$

$f(n) \cdot f(n) \leq c\, g(n) \cdot c\, g(n)$

prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, (f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h)) \Rightarrow (f + g) \in \mathcal{O}(h)$

$f(n) \leq c_1 h(n)$

$g(n) \leq c_2 h(n)$

assume $f, g \in \mathcal{F}$ and $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$ # intro $\forall$ and $\Rightarrow$

Then $\exists c_f \in \mathbb{R}^+, \exists B_f \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_f \Rightarrow f(n) \leq c_f h(n)$

since $f \in \mathcal{O}(h)$

Then $\exists c_g \in \mathbb{R}^+, \exists B_g \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_g \Rightarrow g(n) \leq c_g h(n)$

Pick $c = c_f + c_g$. Then $c \in \mathbb{R}^+$

Pick $B = B_f + B_g$. Then $B \in \mathbb{N}$ and $B \geq \max(B_f, B_g)$

assume $n \in \mathbb{N}$ and $n \geq B$ # to intro $\forall$ and $\Rightarrow$.

Then $(f+g)(n) = f(n) + g(n)$

$\leq c_f h(n) + c_g h(n)$

since $n \geq B_f$ and $n \geq B_g$ and $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$.

$= (c_f + c_g) h(n)$

$= c\, h(n)$ # $c = c_f + c_g \in \mathbb{R}^+$

Then $\exists$...

Then $\forall n \in \mathbb{N}, n \geq B \Rightarrow (f+g)(n) \leq c\, h(n)$ # intro $\forall, \Rightarrow$.

Then $(f+g) \in \mathcal{O}(h)$ # fits defn

Then $\forall f, g \in \mathcal{F}, f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h) \Rightarrow (f+g) \in \mathcal{O}(h)$.

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prove or disprove:

$\mathcal{F} = \{f : \mathbb{N} \mapsto \mathbb{R}^{\geq 0}\}$

$\forall f, g \in \mathcal{F}, f \in \mathcal{O}(g) \Rightarrow f \in \mathcal{O}(g \cdot g)$ disprove

$f(n) \leq c\, g(n)$

$f(n) \leq c\, g(n) \cdot g(n)$

try: $f(n) = g(n) = \frac{1}{2}$

$\exists f, g \in \mathcal{F},\ f \in \Theta(g) \wedge f \notin \Theta(g \cdot g)$.

Pick $f = g = \frac{1}{n+1}$. Then $f, g \in \mathcal{F}$ # not hard to show.

also $f \in \Theta(g)$. # Pick $c = 1$.

Must show $\forall c_1 \in \mathbb{R}^+, \forall B_1 \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B_1 \wedge f(n) > c_1 g(n) \cdot g(n)$

Assume $c_1 \in \mathbb{R}^+$, $B_1 \in \mathbb{N}$ # to introduce $\forall$.

Pick $n = \lceil c \rceil + B$. Then $n \in \mathbb{N}$ and $n \geq B$.

Then $f(n) = \frac{1}{n+1}$

$> c\, g(n) \cdot g(n) = \frac{c}{(n+1)^2}$

$\frac{1}{n+1} > \frac{c}{(n+1)^2}$ if $n \geq c$.

$\frac{1}{n+1} > \frac{c}{(n+1)^2}$ if $n+1 > c$ $\Leftarrow n \geq c$

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Notes