CSC165, Winter 2013
Assignment 2

sample solutions

1. **Prove or disprove:** $\forall x, y \in \mathbb{R}, x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$.

   Sample solution: The statement is true. It says that if you have one real number less than another, then there is another real number strictly between them. The way to prove this is to construct z in terms of x and y. One way to do this is to make z the midpoint.

   Assume x, y are real numbers. # in order to introduce $\forall$

   Assume $x < y$. # in order to introduce $\Rightarrow$

   Pick $z = (x + y)/2.0$. Then $z \in \mathbb{R}$. # $\mathbb{R}$ closed under $+, /2.0$
   Then $z < y$. # Since $x < y \Rightarrow x + y < y + y \Rightarrow (x + y)/2.0 < (y + y)/2.0$.
   Also $x < z$. # Since $x < y \Rightarrow x + x < x + y \Rightarrow (x + x)/2.0 < (x + y)/x.0$
   So $x < z \wedge z < y$. # introduced conjunction.
   So $\exists z \in \mathbb{R}, x < z \wedge z < y$. # introduced $\exists$

   Then $x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$. # introduced $\Rightarrow$

   Then $\forall x, y \in \mathbb{R}, x < y \Rightarrow (\exists z \in \mathbb{R}, x < z \wedge z < y)$. # introduced $\forall$

2. **Prove or disprove:** $\forall m, n \in \mathbb{N}, m < n \Rightarrow (\exists k \in \mathbb{N}, m < k \wedge k < n)$

   Sample solution: The claim is false. To disprove it, just find a pair of consecutive natural numbers, and observe that there is no natural number between them (since they increment by 1). I prove the negation of the original statement.

$$\exists m, n \in \mathbb{N}, m < n \wedge (\forall k \in \mathbb{N}, m \geq k \vee k \geq n)$$

   Pick $m = 1, n = 2$. Then $m, n \in \mathbb{N}$. # 1, 2 are natural numbers

   Then $m < n$. # $1 < 2$.

   Assume $k \in \mathbb{N}$. # in order to introduce $\forall$

   Assume $k > m = 1$. # in order to introduce $\Rightarrow$

   Then $k \geq 2$. # Successor to 1 is $1 + 1 = 2$ in $\mathbb{N}$

   Then $k > m \Rightarrow k \geq 2 = n$. # introduced $\forall$
   Then $k \leq m \vee k \geq n$. # equivalent to $k > m \Rightarrow k \geq n$.

   Then $\forall k \in \mathbb{N}, k \leq m \vee k \geq n$. # introduced $\forall$

   Then $m < n \wedge (\forall k \in \mathbb{N}, m \geq k \vee k \geq n)$. # introduced conjunction

   Then $\exists m, n \in \mathbb{N}, m < n \wedge (\forall k \in \mathbb{N}, m \geq k \vee k \geq n)$. # introduced $\exists$

3. **Prove or disprove:** If n is a natural number that has $n^2 \bmod 11 = 3$, then $n \bmod 11 = 5$. Your proof/disproof must use the definition of $a \bmod b$ on page 11 of the Course Notes.

   Sample solution: The claim is false. To negate it, just produce some perfect square that is equal to 3 mod 11, but whose square root is not equal to 5 mod 11.

$$\exists n \in \mathbb{N}, n^2 \bmod 11 = 3 \wedge n \bmod 11 \neq 5$$

   Pick $n = 6$. Then $n \in \mathbb{N}$. # 6 is a natural number.

   Then $n \bmod 11 = 6 \neq 5$. # $6 = 0 \times 11 + 6 \wedge 11 > 6 \geq 0$.

   # uniqueness of quotient remainder means $6 \bmod 11 = 6$ and not 5

   Then $n^2 = 36 \bmod 11 = 3$. # $36 = 3 \times 11 + 3 \wedge 11 > 3 \geq 0$.

   Then $n^2 \bmod 11 = 3 \wedge n \bmod 11 \neq 5$. # conjunction

   Then $\exists n \in \mathbb{N}, n^2 \bmod 11 = 3 \wedge n \bmod 11 \neq 5$ # introduced $\exists$.

4. **Prove or disprove:** If n is a natural number that has $n \bmod 11 = 5$, then $n^2 \bmod 11 = 3$. Your proof/disproof must use the definition of $a \bmod b$ on page 11 of the Course Notes.

   Sample solution: The claim is true. The main idea is to use the definition to write a natural number that is equal to 5 mod 11 as some product of 11 plus 5, then square this and do some algebra.

   Assume n is a natural number. # in order to introduce $\forall$

   Assume $n \bmod 11 = 5$. # in order to introduce $\Rightarrow$

   Then $\exists q \in \mathbb{N}, n = 11q + 5$. # definition of $n \bmod 11 = 5$.
   Then $n^2 = (11q + 5)^2$. # substitute $n = 11q + 5$.
   Then $n^2 = 121q^2 + 110q + 25$. # expanding
   Then $n^2 = 11(11q^2 + 10q + 2) + 3$. # factoring
   Then $\exists k \in \mathbb{N}, n^2 = 11k + 3 \wedge 11 > 3 \geq 0$.
   # Pick $k = 11q^2 + 10q + 2$, since $11, q, 10, 2 \in \mathbb{N}$ and $\mathbb{N}$ closed under $\times, +$.
   Then $n^2 \bmod 11 = 3$. # by definition

   Then $n \bmod 11 = 5 \Rightarrow n^2 \bmod 11 = 3$. # introduced $\Rightarrow$.

   Then $\forall n \in \mathbb{N}, n \bmod 11 = 5 \Rightarrow n^2 \bmod 11 = 3$. # introduced $\forall$

5. **Prove or disprove:** For all quadruples of positive real numbers w, x, y, z, If $w/x < y/z$ then:

$$\left(\frac{w}{x} < \frac{w+y}{x+z}\right) \wedge \left(\frac{w+y}{x+z} < \frac{y}{z}\right)$$

   Hint: The material on inequalities on page 12 of the Course Notes may be helpful.

   Sample solution: The claim is true. The idea is to transform the inequality in the antecedent, $w/x < y/z$, into the inequalities in the consequent by multiplying both sides by positive numbers (this preserves the inequality). To keep the argument clear, be sure to work in a single direction: from the antecedent to the consequent.

   Assume w, x, y, z are positive real numbers. # in order to introduce $\forall$

   Assume $w/x < y/z$. # in order to introduce $\Rightarrow$

   Then $wz < yx$.
   # multiply both sides of $w/x < y/z$ by $zx \in \mathbb{R}^+$, since $z, x \in \mathbb{R}^+$
   Then $wz + yz < yx + yz$. # add yz to both sides

Then $(w + y)/(x + z) < y/z$. # first part of conjunction
multiply both sides by $1/(z(x + z)) \in \mathbb{R}^+$, since $z, (x + z) \in \mathbb{R}^+$.
Now, add wx to both sides of $wz < yx$ (already established above),
yielding $wx + wz < wx + yx$.
Then $w/x < (w + y)/(x + z)$. # multiply both sides by $1/x(x + z)$
Then $w/x < (w + y)/(x + z) \wedge (w + y)/(x + z) < y/z$. # introduced conjunction

Then $w/x < y/z \Rightarrow w/x < (w + y)/(x + z) \wedge (w + y)/(x + z) < y/z$. # introduced $\Rightarrow$

Then $\forall w, x, y, z \in \mathbb{R}^+, w/x < y/z \Rightarrow w/x < (w + y)/(x + z) \wedge (w + y)/(x + z) < y/z$.
introduced $\forall$

6. **Prove or disprove:** For every pair of positive natural numbers (m, n), if $m \geq n$, then the $\gcd(m, n) = \gcd(n, m - n)$. Your proof/disproof must use the definition of the gcd (Greatest Common Divisor) on page 12 of the Course Notes.

Sample solution: The claim is true. The idea is to use the properties of the gcd from the definition — that it divides both numbers, and is the largest integer that does so. By showing that $\gcd(m, n)$ is a common factor of $(n, m - n)$, and that $\gcd(n, m - n)$ is a common factor of (m, n), you can use the part of the definition about being the **greatest** common factor. . . in two directions.

Assume m, n are positive natural numbers. # in order to introduce $\forall$

Assume $m \geq n$. # in order to introduce $\Rightarrow$

Pick $g_1 = \gcd(m, n), g_2 = \gcd(n, m - n)$. # for convenience
Then $\exists i, j \in \mathbb{N}, m = ig_1, n = jg_1$. # definition of $g_1 = \gcd(m, n)$.
Then $m - n = (i - j)g_1 \wedge i - j \in \mathbb{N}$. # since $m \geq n$ and factoring
Then g_1 divides both n and $m - n$. # by definition of divides
Then $g_2 \geq g_1$. # definition of $g_2 = \gcd(n, m - n)$.
Then $\exists k, l \in \mathbb{N}, n = kg_2, (m - n) = lg_2$. # definition of $g_2 = \gcd(n, m - n)$.
Then $m = (k + l)g_2 \wedge (k + l) \in \mathbb{N}$. # factoring
Then g_2 divides both m and n. # by definition of divides
Then $g_1 \geq g2$. # definition of $g_1 = \gcd(m, n)$.
Then $g_1 \geq g_2 \wedge g_2 \geq g1$. # conjunction
Then $(g_1 > g_2 \vee g_1 = g_2) \wedge (g_2 > g_1 \vee g_2 = g_1)$. # $a \geq b \equiv (a > b \vee a = b)$.
Then $(g_1 > g_2 \wedge g_2 > g_1) \vee g_1 = g_2$. # distribute $\wedge$ over $\vee$.
Then $g_1 = g_2$. # $g_1 > g_2 \wedge g_2 > g_1$ is a contradiction
Then $\gcd(m, n) = \gcd(n, m - n)$. # $g_1 = \gcd(m, n)$, $g_2 = \gcd(n, m - n)$.

Then $m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\Rightarrow$

Then $\forall m, n \in \mathbb{N}, m \geq n \Rightarrow \gcd(m, n) = \gcd(n, m - n)$. # introduced $\forall$