UNIVERSITY OF TORONTO

Faculty of Arts and Science

term test #1, Version 1

CSC165H1S

Date: Tuesday October 10, 6:10–7:00pm

Duration: 50 minutes

Instructor(s): Danny Heap

No Aids Allowed

Name:

utorid:

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Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam.**
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.

Take a deep breath.
This is your chance to show us
How much you've learned.

Good luck!

1. **[7 marks] Translating statements.**

   A **perfect square** is a number that is the product of another number times itself, for example 1, 4, and 9 are perfect squares. A **square-free integer** is an integer that is divisible by no perfect square other than 1.

   Express each of the following statements using predicate logic. No justification is required. Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

   (a) **[3 marks]** 24 is not a square-free integer.

   Solution

   $\exists n \in \mathbb{N}, n > 1 \wedge n^2 \mid 24$

   (b) **[4 marks]** 97 is the largest square-free number less than 101.

   Solution

   $\big[\forall n \in \mathbb{N}, (n > 1 \Rightarrow n^2 \nmid 97)\big] \wedge \big[\forall m \in \mathbb{Z}, \big[(\forall k \in \mathbb{N}, k > 1 \Rightarrow k^2 \nmid m) \wedge m > 97\big] \Rightarrow m \geq 101\big]$

2. **[6 marks] Statements in logic.**

(a) **[3 marks]** Write the truth table for the following formula. No rough work is required.

$$\big((p \Rightarrow q) \wedge r\big) \Leftrightarrow \neg p$$

Solution

p	q	r	$\big((p \Rightarrow q) \wedge r\big) \Leftrightarrow \neg p$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	F

(b) **[3 marks]** Consider the pair of statements:

(1) $\exists n \in \mathbb{N},\ P(n) \Rightarrow Q(n)$ (2) $\exists n \in \mathbb{N},\ P(n) \wedge Q(n)$

Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q. Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

Solution

Let $P(n)$ be the predicate "$n < 0$" and $Q(n)$ be the predicate "$n \geq 0$".

The first statement becomes $\exists n \in \mathbb{N}, n < 0 \Rightarrow n \geq 0$, which is true by vacuous truth (the hypothesis is false for all natural numbers). The second statement becomes $\exists n \in \mathbb{N}, n < 0 \wedge n \geq 0$, which is false, since no natural numbers are less than 0.

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "For every natural number x there is a real number y such that $1650 > xy > 165$."

   (a) **[1 mark]** Translate the above statement into predicate logic.

   Solution

   $\forall x \in \mathbb{N}, \exists y \in \mathbb{R}, xy < 1650 \wedge xy > 165$

   (b) **[1 mark]** Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a > b)$ to $a \leq b$.

   Solution

   $\exists x \in \mathbb{N}, \forall y, xy \geq 1650 \vee xy \leq 165$

   (c) **[4 marks]** Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

   Solution

   Proof. Let $x = 0$. Let $y \in \mathbb{R}$. Then $xy = 0 \leq 165$. ■ □

4. **[5 marks] Proofs (number theory).** Consider the following statement: "If m and n are integers, and 5 divides both m and n, then 5 divides $m - n$."

   (a) **[1 mark]** Translate the above statement into predicate logic.

   Solution

   $\forall m, n \in \mathbb{Z}, 5 \mid m \wedge 5 \mid n \Rightarrow 5 \mid (m - n)$ Also allow the literal, improperly-introduced m and n: $(m, n \in \mathbb{Z} \wedge 5 \mid m \wedge 5 \mid n) \Rightarrow 5 \mid (m - n)$

   (b) **[4 marks]** Prove the above statement using the definition of divisibility:

$$x \mid y : \ \exists k \in \mathbb{Z}, \ y = kx$$

   Do not use any external facts about divisibility.

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

   Solution

   Proof. Let $m, n \in \mathbb{Z}$. Assume $5 \mid m$ and $5 \mid n$, that is $\exists k_1, k_2 \in \mathbb{Z}, m = 5k_1 \wedge n = 5k_2$. Let k_1, k_2 be such values. Let $k_3 = k_1 - k_2$.

   Then $m - n = 5k_1 - 5k_2 = 5(k_1 - k_2) = 5k_3 \quad \blacksquare$. □

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Name:

Question	Grade	Out of
Q1		7
Q2		6
Q3		6
Q4		5
Total		24