UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm 2, Version 1
CSC165H1S

Date: Thursday March 23, 6:10-7:00pm
Duration: 50 minutes
Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam**.
- This examination has **4** questions. There are a total of **9 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- For algorithm analysis questions, you can jump immediately from a step count to an asymptotic bound without proof (e.g., write "the number of steps is $3n + \log n$, which is $\Theta(n)$").

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

1. **[5 marks] Induction.** Prove the following statement using induction on n:

$$\forall m \in \mathbb{Z}^+,\ \forall n \in \mathbb{N},\ n \geq 2 \Rightarrow m^n + 3 \leq (m+1)^n$$

Hint: $(m+1)^{n+1} = (m+1)(m+1)^n$.

2. **[6 marks] Worst-case runtime.** Consider the following algorithm, which takes as input a list of integers.

```
def alg(A):
    n = len(A)
    count = 0
    for i in range(n):              # Loop 1
        if A[i] >= count:
            count = count + 1

    for j in range(count):          # Loop 2
        for k in range(j):          # Loop 3
            print('Counted!')
```

Let $WC(n)$ be the worst-case runtime function of `alg`, where n is the length of the input list A. You can use the following formula in your analysis of $WC(n)$:

$$\forall m \in \mathbb{N},\ \sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

Note: assume the integers stored in A can be arbitrarily large (i.e., don't assume some upper limit on the numbers in A).

(a) Find, with proof, a good asymptotic upper bound (Big-Oh) on $WC(n)$. By "good" we mean that if you prove $WC \in \mathcal{O}(f)$ (where you chose the f), it should be true that $WC \in \Omega(f)$ as well (but don't prove this here).

(b) Describe an input family whose runtime matches the upper bound you proved in part (a). For example, if you proved that $WC(n) \in \mathcal{O}(n)$, for this part you should describe an input family whose runtime is $\Theta(n)$.

Only a description of the input family is necessary; you do **not** need to analyse the running time of `alg` on your chosen input family.

3. **[4 marks] Best-case runtime.** Let $BC(n)$ be the best-case running time of the algorithm `alg` from Question 2. Prove that $BC(n) \in \mathcal{O}(n)$, where n represents the length of the input list. You may assume that $n > 0$ for this analysis.

4. **[5 marks] Properties of Big-Oh.** For all functions $f, g \in \mathbb{N} \to \mathbb{R}^{\geq 0}$, we define their *sum function*, denoted $f + g$, to be the following function:

$$(f + g)(n) = f(n) + g(n) \quad \text{for all } n \in \mathbb{N}.$$

Prove that for all functions $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $g_1 \in \mathcal{O}(f_1)$ and $g_2 \in \mathcal{O}(f_2)$, then $g_1 + g_2 \in \mathcal{O}(f_1 + f_2)$.

Reminder: you may not use any properties of Big-Oh in this question. You should use the definition of Big-Oh:

$$g \in \mathcal{O}(f): \quad \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n), \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

Name:

Question	Grade	Out of
Q1		5
Q2		6
Q3		4
Q4		5
Total		20