UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm 1, Version 3
CSC165H1S

Date: Friday February 10, 2:10-3:00pm
Duration: 50 minutes
Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam**.
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely. Provide justification unless explicitly asked not to.
- All formulas must have negations applied directly to propositional variables or predicates.
- In your proofs, you may always use definitions of predicates from the course. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- For algorithm analysis questions, you can jump immediately from a step count to an asymptotic bound without proof (e.g., write "the number of steps is $3n + \log n$, which is $\Theta(n)$").

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

1. **[6 marks] Statements in logic.**

   (a) **[3 marks]** Write the truth table for the following formula. No rough work is required.

   $$\big((p \Leftrightarrow q) \land \neg r\big) \Rightarrow r$$

   Hint: use vacuous truth to quickly find some rows where the formula is true.

   (b) **[3 marks]** Consider the pair of statements:

   $$(1) \quad \big(\exists n \in \mathbb{N},\ P(n)\big) \land \big(\exists m \in \mathbb{N}, Q(m)\big) \qquad\qquad (2) \quad \exists n \in \mathbb{N},\ P(n) \land Q(n)$$

   Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q.

   Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

2. **[7 marks] Translating statements**.

Let $x \in \mathbb{N}$. We say that x is a **twin prime** if and only if both x and $x + 2$ are prime. For example, 5 is a twin prime, because both 5 and 7 are prime. Note that 7 is *not* a twin prime, since 9 is not prime.

Express each of the following statements using predicate logic. No justification is required.

Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) For every twin prime p, $2p + 1$ is also a twin prime.

(b) There are infinitely many numbers that are *not* a twin prime.
Hint: it may be easier to first express "n is not a twin prime."

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "There exists a real number x such that x is less than 3 and for every real number y, $x + y^2 > 25$."

   (a) **[1 mark]** Translate the above statement into predicate logic.

   (b) **[1 mark]** Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a > b)$ to $a \leq b$ and $\neg(a < b)$ to $a \geq b$.

   (c) **[4 marks]** Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!
   Hint: use the fact that $\neg p \vee q$ is equivalent to $p \Rightarrow q$ to rewrite the negation into an implication.
   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be graded if your proof is not completely correct.

Discussion.

Proof.

4. **[5 marks] Proofs (number theory).** Consider the following statement: "For all two integers x and y, if x and y are both divisible by 3, then $x^2 + 2y^2$ is divisible by 3."

   (a) **[1 mark]** Translate the above statement into predicate logic.

   (b) **[4 marks]** Prove the above statement using the definition of divisibility:

$$x \mid y : \quad \exists k \in \mathbb{Z},\ y = kx$$

   Do not use any external facts about divisibility.
   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Discussion.

Proof.

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
Total		24