Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements using the definition of Big-Oh.
- Investigate properties of Big-Oh of some common functions.

Note: in Big-Oh expressions, it will be convenient to just write down the "body" of the functions rather than defining named functions all the time. We'll always use the variable n to represent the function input, and so when we write "$n \in \mathcal{O}(n^2)$," we really mean "the functions defined as $f(n) = n$ and $g(n) = n^2$ satisfy $f \in \mathcal{O}(g)$."

As a reminder, here is the formal definition of what "g is Big-Oh of f" means:

$$g \in \mathcal{O}(f) : \; \exists c, n_0 \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq cf(n)$$

1. **Comparing polynomials.** Our first step in comparing different types of functions is looking at different powers of n. Consider the following statement, which generalizes the idea that $n \in \mathcal{O}(n^2)$:

$$\forall a, b \in \mathbb{R}^+, \; a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)$$

(a) Rewrite the above statement, but with the definition of Big-Oh expanded.

Solution

$$\forall a, b \in \mathbb{R}^+, \; a \leq b \Rightarrow \left(\exists c, n_0 \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow n^a \leq cn^b\right)$$

(b) Prove the above statement. **Hint**: you can actually pick c and n_0 to both be 1, and have the proof work.

Solution

Proof. Let $a, b \in \mathbb{R}$, and assume $a \leq b$. Let $c = 1$ and $n_0 = 1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We want to prove that $n^a \leq n^b$.

We can start with our assumption that $a \leq b$ and calculate:

$$\begin{aligned} a &\leq b \\ n^a &\leq n^b && \text{(since } n \geq 1\text{)} \\ n^a &\leq cn^b && \text{(since } c = 1\text{)} \end{aligned}$$

□

2. **Comparing logarithms.** One slight oddness about the definition of Big-Oh is that it treats all logarithmic functions "the same." Your task in this question is to investigate this, by proving the following statement:

$$\forall a, b \in \mathbb{R}^+, \; a > 1 \land b > 1 \Rightarrow \log_a n \in \mathcal{O}(\log_b n)$$

We won't ask you to expand the definition of Big-Oh, but if you aren't quite sure, then we highly recommend doing so before attempting even your rough work!

Hint: look up the "change of base rule" for logarithms, if you don't quite remember it!

Solution

Proof. Let $a, b \in \mathbb{R}^+$. Assume that $a > 1$ and $b > 1$. Let $n_0 = 1$, and let $c = \dfrac{1}{\log_b a}$. [Since $a, b > 1$, we know that $c > 0$.] Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We want to show that $\log_a n \leq c \cdot \log_b n$.

The *change of base rule* tells us the following:

$$\forall a, b, x \in \mathbb{R}^+,\ a \neq 1 \land b \neq 1 \Rightarrow \log_a x = \frac{\log_b x}{\log_b a}$$

(Note that when the bases are equal to 1, $\log_a x$ is undefined when $x \neq 1$.)

Using this rule, we can write:

$$\begin{aligned}
\log_a n &= \frac{\log_b n}{\log_b a} \\
&= \frac{1}{\log_b a} \log_b n \\
&= c \cdot \log_b n
\end{aligned}$$

Since we've proved that $\log_a n = c \cdot \log b_n$, we can conclude that $\log_a n \leq c \cdot \log_b n$.

[*Note*: we didn't use the assumption that $n \geq 1$ in this proof.] $\square$

3. **Sum of functions.** Now let's look at one of the most important properties of Big-Oh: how it behaves when adding functions together. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ (i.e., f and g are two functions that take natural numbers and return non-negative real numbers). We can define the **sum of** f **and** g as the function $f + g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ such that

$$\forall n \in \mathbb{N},\ (f + g)(n) = f(n) + g(n).$$

For example, if $f(n) = 2n$ and $g(n) = n^2 + 3$, then $(f + g)(n) = 2n + n^2 + 3$.

Consider the following statement:

$$\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)$$

In other words, if g is Big-Oh of f, then $f + g$ is no bigger than just f itself, asymptotically speaking.[1]

Your task for this question is to prove this statement. Keep in mind this is an implication: you're going to *assume* that $g \in \mathcal{O}(f)$, and you want to *prove* that $f + g \in \mathcal{O}(f)$. It will likely be helpful to write out the full statement (with the definition of Big-Oh expanded), and use subscripts to help keep track of the variables.

Solution

Here's the full statement, with the definitions expanded:

$$\begin{aligned}\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ &\Big(\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n)\Big) \Rightarrow \\ &\Big(\exists c_1, n_1 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_1 \Rightarrow f(n) + g(n) \leq c_1 f(n)\Big)\end{aligned}$$

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume that $g \in \mathcal{O}(f)$, i.e., there exist $n_0, c \in \mathbb{R}^+$ such that for all natural numbers n, if $n \geq n_0$ then $g(n) \leq cf(n)$. We want to prove that $f + g \in \mathcal{O}(f)$.

Let $n_1 = n_0$, and $c_1 = c + 1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_1$. We want to prove that $f(n) + g(n) \leq c_1 f(n)$.

Since $n \geq n_1 = n_0$, by our assumption we know that $g(n) \leq cf(n)$. So then:

$$\begin{aligned} g(n) &\leq cf(n) \\ f(n) + g(n) &\leq f(n) + cf(n) \\ f(n) + g(n) &\leq (c + 1)f(n) \\ f(n) + g(n) &\leq c_1 f(n) \end{aligned}$$

$\square$

[1] This statement is quite similar to ones about divisibility, and in particular Question 1 on Problem Set 2.