

Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements using the definition of Big-Oh.
- Investigate properties of Big-Oh of some common functions.

Note: in Big-Oh expressions, it will be convenient to just write down the “body” of the functions rather than defining named functions all the time. We’ll always use the variable n to represent the function input, and so when we write “ $n \in \mathcal{O}(n^2)$,” we really mean “the functions defined as $f(n) = n$ and $g(n) = n^2$ satisfy $f \in \mathcal{O}(g)$.”

As a reminder, here is the formal definition of what “ g is Big-Oh of f ” means:

$$g \in \mathcal{O}(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n)$$

1. **Comparing polynomials.** Our first step in comparing different types of functions is looking at different powers of n . Consider the following statement, which generalizes the idea that $n \in \mathcal{O}(n^2)$:

$$\forall a, b \in \mathbb{R}^+, a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)$$

- (a) Rewrite the above statement, but with the definition of Big-Oh expanded.

- (b) Prove the above statement. **Hint:** you can actually pick c and n_0 to both be 1, and have the proof work.

2. **Comparing logarithms.** One slight oddness about the definition of Big-Oh is that it treats all logarithmic functions “the same.” Your task in this question is to investigate this, by proving the following statement:

$$\forall a, b \in \mathbb{R}^+, a > 1 \wedge b > 1 \Rightarrow \log_a n \in \mathcal{O}(\log_b n)$$

We won't ask you to expand the definition of Big-Oh, but if you aren't quite sure, then we highly recommend doing so before attempting even your rough work!

Hint: look up the “change of base rule” for logarithms, if you don't quite remember it!

3. **Sum of functions.** Now let's look at one of the most important properties of Big-Oh: how it behaves when adding functions together. Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ (i.e., f and g are two functions that take natural numbers and return non-negative real numbers). We can define the **sum of f and g** as the function $f + g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ such that

$$\forall n \in \mathbb{N}, (f + g)(n) = f(n) + g(n).$$

For example, if $f(n) = 2n$ and $g(n) = n^2 + 3$, then $(f + g)(n) = 2n + n^2 + 3$.

Consider the following statement:

$$\forall f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}, g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)$$

In other words, if g is Big-Oh of f , then $f + g$ is no bigger than just f itself, asymptotically speaking.¹

Your task for this question is to prove this statement. Keep in mind this is an implication: you're going to *assume* that $g \in \mathcal{O}(f)$, and you want to *prove* that $f + g \in \mathcal{O}(f)$. It will likely be helpful to write out the full statement (with the definition of Big-Oh expanded), and use subscripts to help keep track of the variables.

¹ This statement is quite similar to ones about divisibility, and in particular Question 1 on Problem Set 2.