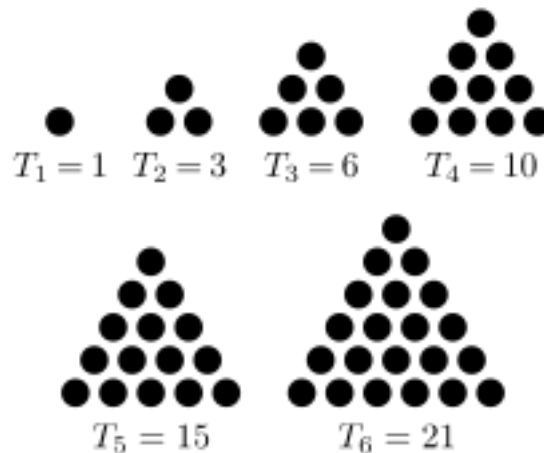


Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

1. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with n marbles on each side, a total of $\sum_{i=1}^n i$ marbles will be required.



In lecture, we proved that $\sum_{i=1}^n i = n(n+1)/2$. For each $n \in \mathbb{N}$, let $T_n = n(n+1)/2$; these numbers are usually called the *triangular numbers*. Use induction to prove that

$$\forall n \in \mathbb{N}, \sum_{j=0}^n T_j = \frac{n(n+1)(n+2)}{6}$$

Solution

Let us start by defining the predicate

$$P(n) : \sum_{j=0}^n T_j = \frac{n(n+1)(n+2)}{6}$$

We need to prove that $\forall n \in \mathbb{N}, P(n)$.

Proof. **Base case:** let $n = 0$. We want to prove $P(0)$. Then we can calculate:

$$\begin{aligned} \sum_{j=0}^n T_j &= \sum_{j=0}^0 T_j \\ &= T_0 \\ &= \frac{0(0+1)}{2} \\ &= 0 \end{aligned}$$

And also $\frac{0(0+1)(0+2)}{6} = 0$.

Induction step: Let $k \in \mathbb{N}$ and assume $P(k)$, i.e., that $\sum_{j=0}^k T_j = k(k+1)(k+2)/6$. We want to prove $P(k+1)$,

i.e., that $\sum_{j=0}^{k+1} T_j = (k+1)(k+2)(k+3)/6$.

We'll calculate starting from the left side and show that it equals the right side.

$$\begin{aligned}
 \sum_{j=0}^{k+1} T_j &= \left(\sum_{j=0}^k T_j \right) + T_{k+1} \\
 &= \frac{k(k+1)(k+2)}{6} + T_{k+1} && \text{(by our assumption of } P(k)) \\
 &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} && \text{(by the definition of } T_{k+1}) \\
 &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{6} \\
 &= \frac{(k+1)(k+2)(k+3)}{6}
 \end{aligned}$$

□

2. **Induction (inequalities).** Consider the statement:

For every positive real number x and every natural number n , $(1 + x)^n \geq (1 + nx)$.

We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$$

Notice that in this statement, there are two universally-quantified variables: n and x . Prove the statement is true using the following approach:

- (a) Use the standard proof structure to introduce x .
- (b) When proving the $(\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx)$, do induction on n .¹

Solution

Proof. Let $x \in \mathbb{R}^+$. We'll prove that for all $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$ by induction.

Base case: Let $n = 0$.

Then $(1 + x)^0 = 1$ and $1 + nx = 1$. So then $(1 + x)^n \geq 1 + nx$.

Induction step: Let $k \in \mathbb{N}$, and assume that $(1 + x)^k \geq 1 + kx$. We want to prove that $(1 + x)^{k+1} \geq 1 + (k + 1)x$.

We'll start with the quantity on the left, and show that it's \geq the quantity on the right.

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)^k(1 + x) \\ &\geq (1 + kx)(1 + x) && \text{(by our assumption)} \\ &= 1 + kx + x + kx^2 \\ &\geq 1 + kx + x && \text{(since } kx^2 \geq 0) \\ &= 1 + (k + 1)x \end{aligned}$$

□

¹ Your predicate $P(n)$ that you want to prove will also contain the variable x – that's okay, since when we do the induction proof, x has already been defined.

3. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N}, n \leq 2^n$ using induction.

- (a) First, use trial and error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!

$$\forall n \in \mathbb{N}, n \geq \text{_____} \Rightarrow 30n \leq 2^n$$

Solution

$$\forall n \in \mathbb{N}, n \geq 8 \Rightarrow 30n \leq 2^n.$$

- (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!

Solution

Proof. **Base case:** let $n = 8$.

Then $30n = 240$, and $2^n = 256$. So $30n \leq 2^n$.

Induction step: let $k \in \mathbb{N}$. Assume that $k \geq 8$, and that $30k \leq 2^k$. We want to prove that $30(k+1) \leq 2^{k+1}$.

Since $8 \leq k$, we know that $256 \leq 2^k$ (raising 2 to the power of either side). Our assumption tells us that $30k \leq 2^k$. Adding these two inequalities yields:

$$30k + 256 \leq 2^k + 2^k$$

$$30k + 256 \leq 2^{k+1}$$

$$30k + 30 \leq 2^{k+1}$$

(since $30 \leq 256$)

$$30(k+1) \leq 2^{k+1}$$

□