Learning Objectives

By the end of this worksheet, you will:

- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

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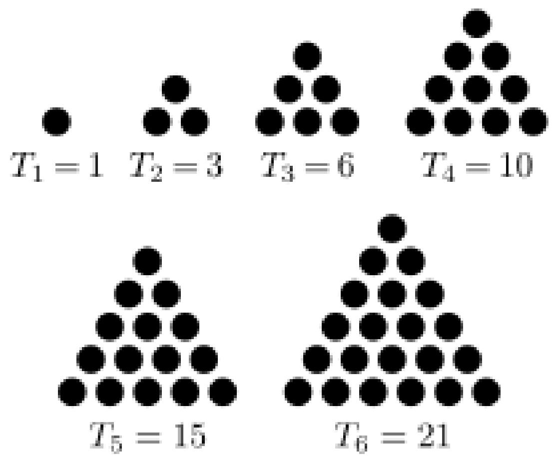
1. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with n marbles on each side, a total of $\sum_{i=1}^{n} i$ marbles will be required.



In lecture, we proved that $\sum_{i=1}^{n} i = n(n+1)/2$. For each $n \in \mathbb{N}$, let $T_n = n(n+1)/2$; these numbers are usually called the *triangular numbers*. Use induction to prove that

$$\forall n \in \mathbb{N}, \ \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$$

2. **Induction (inequalities).** Consider the statement:

   For every positive real number x and every natural number n, $(1+x)^n \geq (1+nx)$.

   We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ (1+x)^n \geq 1+nx$$

   Notice that in this statement, there are two universally-quantified variables: n and x. Prove the statement is true using the following approach:

   (a) Use the standard proof structure to introduce x.
   (b) When proving the $\left(\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx\right)$, do induction on n.[1]

[1] Your predicate $P(n)$ that you want to prove will also contain the variable x – that's okay, since when we do the induction proof, x has already been defined.

3. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N},\ n \leq 2^n$ using induction.

   (a) First, use trial an error to fill in the blank to make the following statement true – try finding the *smallest natural number* that works!

   $$\forall n \in \mathbb{N}, n \geq \underline{\hspace{3cm}} \Rightarrow 30n \leq 2^n$$

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!