— Sample T1 solutions tomorrow...

CSC165 fall 2017

Mathematical expression: induction: different starting points, etc.

Danny Heap
csc16517f@cs.toronto.edu
BA4270 (behind elevators)
Web page:
http://www.teach.cs.toronto.edu/~heap/165/F17/
416-978-5899

Using Course notes: more Induction

Outline

notes

review induction parts

- claim

  what we want to prove, usually a univ. quantified pred. of $\mathbb{N}$. often name as a convenience, e.g. $P(n)$

  $\forall n \in \mathbb{N}, P(n)$

  - base case — choose a case (number) to start at — so far 0
  - inductive step — $\forall k \in \mathbb{N}, \underbrace{P(k) \Rightarrow P(k+1)}$

Computer Science
UNIVERSITY OF TORONTO

more domino logistics....

$7^0 \equiv 1 \pmod 6 \quad 7^1 \equiv 1 \pmod 6 \quad 7^2 \equiv 1 \pmod 6 \quad 7^3 \equiv 1 \pmod 6 \quad 7^4 \equiv 1 \pmod 6 \quad 7^5 \equiv 1 \pmod 6 \quad \cdots$ "etc."

$\overbrace{7^0 \equiv 1 \pmod 6 \quad 7^1 \equiv 1 \pmod 6}^{\text{inconvenient}} \quad 7^2 \equiv 1 \pmod 6 \quad 7^3 \equiv 1 \pmod 6 \quad 7^4 \equiv 1 \pmod 6 \quad 7^5 \equiv 1 \pmod 6 \quad \cdots$ "etc."

$2n + 1 < 2^n$ $\qquad$ $P(n): n > 2 \Rightarrow 2n+1 < 2^n$

Want to prove: $\forall n \in \mathbb{N}, P(n)$

Proof (by induction)

Base Case $2 \times 3 + 1 = 7 < 8 = 2^3$. So $P(3)$ is true ✓

Inductive Step Let $n \in \mathbb{N}$. Assume $P(n)$: $n \geq 3 \Rightarrow 2n+1 < 2^n$. So assume $n \geq 3$. Want to show that $P(n+1)$ follows: that $2(n+1)+1 < 2^{n+1}$.

So, $2(n+1)+1 = 2n+2+1 = \underline{2n+1} + 2$

$< 2^n + 2$ (IH) $< 2^n + 2^n$ $\quad (n \geq 3,\ 2^n \geq 8 > 2)$

$= 2^{n+1}$ ■

$\forall n \in \mathbb{N}, 3^n \geq n^3$

~~P(n): ...~~

Proof (induction)

Base cases $3^0 = 1 \geq 0 = 0^3$; $3^1 = 3 \geq 1 = 1^3$; $3^2 = 9 \geq 8 = 2^3$;

$3^3 = 27 \geq 27 = 3^3$. So $P(0)$, $P(1)$, $P(2)$ & $P(3)$

are true.

Induction Step Let $k \in \mathbb{N}$. Assume $k \geq 3$

and $P(k)$: $3^k \geq k^3$. I must show that

$3^{k+1} \geq (k+1)^3$.

$$3^{k+1} = 3 \times 3^k = 3^k + 3^k + 3^k$$

$$\geq k^3 + k^3 + k^3 \qquad \text{(IH)}$$

$$\geq k^3 + 3k^2 + 3k^2 \qquad (k \geq 3 \Rightarrow k^2 \geq 3k)$$

$k^3 \geq 3k^2$

$\forall n \in \mathbb{N}, 3^n \geq n^3$

$$k^3 + 3k^2 + 3k^2 \geq k^3 + 3k^2 + 9k \qquad \left(\begin{array}{c} k \geq 3 \\ 3k \cdot k \geq 9k \end{array}\right)$$

$$= k^3 + 3k^2 + 3k + 6k$$

$$\geq k^3 + 3k^2 + 3k + 1 \qquad \left(\begin{array}{c} k \geq 3 \\ 6k \geq 18 \geq 1 \end{array}\right)$$

$$= (k+1)^3$$

$\forall x, y \in \mathbb{N}, \forall n \in \mathbb{N}, x - y \quad x^n - y^n$: $P(n)$

order of introductions...

Proof

Let $x, y \in \mathbb{N}$.

I prove by induction that $\forall n \in \mathbb{N}$, $x - y \mid x^n - y^n$

Base cases $x^0 - y^0 = 1 - 1 = 0$ so $x - y \mid 0$.

Induction step Let $k \in \mathbb{N}$. Assume $x - y \mid x^k - y^k$.

that is $\exists h_k \in \mathbb{Z}$, $x^k - y^k = h_k(x - y)$. Let

$h_{k+1} = x h_k + y^k$.

Then $x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x - y)$

$\forall x, y \in \mathbb{N}, \forall n \in \mathbb{N}, x - y \quad x^n - y^n$

order of introductions...

$$x(x^k - y^k) + y^k(x-y) = x\,h_k(x-y) + y^k(x-y)$$

$$= h_{k+1}(x-y)$$

every set with n elements has 2^n subsets

more order of introductions...

$\{a, b, c\} = S$ $\quad |S| = 3$

$\mathcal{P}(S)$

$\left| \{ \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \{\} \} \right| = 8$

$P(n)$: Every set S with $|S| = n$ has 2^n subset.

Proof (induction).

Base case $|\{\}| = 0$ and it has $1 = 2^0$ subsets

$\{\}$.

Induction step Let $k \in \mathbb{N}$. Assume $P(k)$: Every set of size k has 2^k subsets. Let S be an arbitrary set with $|S| = k+1$. Want to show that S has 2^{k+1} subsets

every set with n elements has 2^n subsets

more order of introductions...

Since S has $k+1 \geq 1$ elements, we can let one be x. Partition subsets of S into subset that contain element x and those that do not. Note the subset of S that do not contain x are exactly the subsets of $|S \setminus \{x\}| = k$, so there are 2^k of them (IH).

The subsets that do contain x are in 1-1 corresp with subsets of $S \setminus \{x\}$, simply by adding/removing element x. So there are 2^k of them. In all there are $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ $\blacksquare$

Notes

want $2(n+1)+1 < 2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$

have (assume) $n \geq 3$ and $2^n \geq 2n+1$

IH

$2n + 2 + 1 \quad \ldots\ldots \quad 2^n + 2^n \quad n \geq 3$

want $3^{k+1} \geq (k+1)^3$

<u>assume</u> $k \geq 3$ <u>and</u> $3^k \geq k^3$

$$3^{k+1} = 3 \times 3^k = 3^k + 3^k + 3^k$$

$$\geq k^3 + k^3 + k^3$$

$$\geq \qquad \geq \qquad \geq$$

$$9k$$

$$3k + 6k$$

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

$k \geq 3$

$3k^2 \geq 3k$

$k^3 \geq 3k^2$

$k \geq 3$

$k^2 \geq 3 \cdot 3 = 9$

$k^3 \geq 9k$

* want to show $3k^3 \geq (k+1)^3$