Learning Objectives

By the end of this worksheet, you will:

- Prove statements about sets using induction.

So far, we have only used induction to prove statements about numbers. In CSC236, you'll see many more applications of the principle of induction beyond just numbers, but in this tutorial you'll get a taste of what's to come by learning how to use induction on another familiar entity: sets.

1. **A first example** Consider the following statement: "Every finite set S has exactly $\frac{|S|(|S|-1)}{2}$ subsets of size 2."[1] This is a universally-quantified statement, but given an arbitrary set S, it isn't obvious how to make an argument for its number of subsets. But it isn't obvious how to use induction, either! After all, we only know (right now) how to apply induction to prove statements about the natural numbers.

   Here's the big insight into how to apply induction to (finite) sets: every set has a natural corresponding natural number, its *size*. We can rewrite the initial statement to emphasize the size of the sets, as

   $$\text{For every } n \in \mathbb{N}, \text{ every set of size } n \text{ has } \frac{n(n-1)}{2} \text{ subsets of size 2.}$$

   If we define the predicate $P(n)$: "every set S of size n has $\frac{n(n-1)}{2}$ subsets of size 2," then this statement is exactly the kind of statement we can prove using induction!

   (a) Check your understanding: write down, in English, what $P(0)$ means. (This is the base case of the induction proof.)

   Solution

   Every set of size 0 has 0 subsets of size 2.

   (b) Prove $P(0)$. We have started the proof for you.

   Proof. Base case: let $n = 0$. Let S be a set, and assume S has size 0.

   Solution

   We want to prove that S has 0 subsets of size 2.

   Since S has no elements, it doesn't have any subsets of size 2 (in fact, its only subset is itself, which has size 0).

   □

   (c) Now we'll prove the induction step, in a series of part. Please read through and complete the following proof.

   Proof. **Induction step**. Let $k \in \mathbb{N}$, and assume $P(k)$. That is, assume that:

   [*Write down, in English, the induction hypothesis (what $P(k)$ means).*]

   Solution

   Every subset of size k has exactly $\frac{k(k-1)}{2}$ subsets.

   We want to prove $P(k+1)$, that is,

   [*Write down, in English, what you want to prove (what $P(k+1)$ means).*]

[1] For example, a set containing 4 elements has $\frac{4 \cdot 3}{2} = 6$ subsets of size 2.

Solution

Every subset of size $k+1$ has exactly $\frac{(k+1)k}{2}$ subsets.

Let S be a size of $k+1$, and let the elements in the set be $\{s_1, s_2, \ldots, s_k, s_{k+1}\}$.

The **key idea** of most proofs by induction on sets is that you can take a set of size $k+1$, and split it up into a single element and another set of size k.[2] Let $S' = \{s_1, s_2, \ldots, s_k\}$, so that $S = S' \cup \{s_{k+1}\}$. Note that S' has size k. Now, we'll treat s_{k+1} as "special," and us it to divide up the possible subsets of S in a really nice way!

Part 1: counting subsets of size 2 that contain s_{k+1}.

Solution

We'll prove that the number of subsets of S of size 2 that contain s_{k+1} is exactly k.

Every subset of S of size 2 that contains s_{k+1} must contain exactly one element from S'; there are k choices of elements from S' (since $|S'| = k$), and so k subsets of S of size 2 that contain s_{k+1}.

Part 2: counting subsets of size 2 that don't contain s_{k+1}.

[*Use the induction hypothesis to determine the number of subsets of S of size 2 that don't contain s_{k+1}.*][3]

Solution

Every subset of size 2 of S that doesn't contain s_{k+1} must contain 2 of the elements $\{s_1, \ldots, s_k\}$. That is, these subsets are exactly the subsets of size 2 of S'. Since S' has size k, the induction hypothesis tells us that S' has exactly $\frac{k(k-1)}{2}$ subsets of size 2.

Part 3: putting the counts together.

[*Finish off this proof by adding up your results from Parts 1 and 2.*]

Solution

By combining the two counts from Parts 1 and 2, the total number of subsets of size 2 of S is

$$k + \frac{k(k-1)}{2} = \frac{2k + k(k-1)}{2} = \frac{k(k+1)}{2}$$

□

2. **Extending your work.** Use the same technique from the previous question to prove the following statement: "Every finite set S has exactly $\frac{|S|(|S|-1)(|S|-2)}{6}$ subsets of size 3." You can (and should) use the statement you used in the previous question as an external fact in your proof.

Solution

The only difference between this proof and the previous one is that here, the subsets of size 3 that contain the "last" element s_{k+1} correspond to the subsets of **size 2** from a set of size k. That's why you need to use what you proved in the previous question to count the number of subsets of size 3 in this part.

[2] This is the exact same idea as taking a summation with range 1 to $(k+1)$, and splitting it up into a summation with range 1 to k, plus the $(k+1)$-th term.

[3] Note that the induction hypothesis only applies to sets of size k; do you have a set of size k in this proof to work with?