

Learning Objectives

By the end of this worksheet, you will:

- Practice writing proofs and disproofs of statements.
 - Prove statements using the techniques of simple induction and contradiction.
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1. **Induction.** Consider the following statement:

$$\forall n \in \mathbb{N}, n \leq 2^n$$

- (a) Suppose we want to prove this statement using induction. Write down the full statement we'll prove (it should be an **AND** of the base case and induction step). Consult your notes if you aren't sure about this!

- (b) Prove the statement using induction. We strongly recommend reviewing the induction proof template from lecture before working on a proof here.

Hint: $2^{k+1} = 2^k + 2^k$.

2. **A proof by contradiction.** Consider the following statement:¹

$$\forall x, y \in \mathbb{Z}^+, x^2 - y^2 \neq 1$$

Prove this statement using a proof by contradiction. Make sure you clearly identify the following two things (which are specific to proofs by contradiction):

- (a) What you're assuming (the negation of the original statement).
- (b) The contradiction that is made (this should be the last line of the proof).

For this question, you can use external facts you have already seen in this course (in previous worksheets, for example).

¹ In English, we would say that “the equation $x^2 - y^2 = 1$ has no positive integer solutions.”

3. **Linear combinations and divisibility.** Let $x, y \in \mathbb{Z}$. A **linear combination** of x and y is a number that can be written in the form $px + qy$, where p and q are some integers.

(a) Show how to define the predicate $LinComb(x, y, z)$, which says that z is a linear combination of x and y , where $x, y, z \in \mathbb{Z}$.

(b) Translate the following statement into predicate logic:

For every pair of even integers x and y , every linear combination of x and y is also even.

Use the divisibility predicate in the form $2 \mid x$ to express the fact that x is even.

(c) Prove the statement from part (b). Do not use any external facts about divisibility in your proof.

- (d) *Extra*. Generalize the statement from part (b) by replacing “is even” by “is divisible by d ” (for an arbitrary $d \in \mathbb{N}$). Then, prove it!