Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

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1. **A direct proof**. Recall that we say an integer n is **odd** if and only if $\exists k \in \mathbb{Z},\ n = 2k - 1$. Using the technique from lecture, prove the following statement:

   For every pair of odd integers, their product is odd.

   Be sure to translate the statement into predicate logic. You can use the predicate $Odd(n)$: "n is odd" in your formula without expanding the definition, but you'll need to use the definition in your proof.

2. **An incorrect proof.** Consider the following claim:

   For every even integer m and odd integer n, $m^2 - n^2 = m + n$.

   (a) Using the predicates $Even(n)$ and $Odd(n)$ (which return whether an integer n is even or odd, respectively), express the above statement using the notation of symbolic logic.

   (b) The following argument was submitted as a proof of the statement:

   Proof. Let m and n be arbitrary integers, and assume m is even and n is odd. By the definition of even, $\exists k \in \mathbb{Z},\ m = 2k$; by the definition of odd, $\exists k \in \mathbb{Z},\ n = 2k - 1$. We can then perform the following algebraic manipulations:

$$\begin{aligned} m^2 - n^2 &= (2k)^2 - (2k-1)^2 \\ &= 4k^2 - 4k^2 + 4k - 1 \\ &= 4k - 1 \\ &= 2k + (2k - 1) \\ &= m + n \end{aligned}$$

   □

The given argument is not a correct proof. What is the flaw?[1]

3. **Comparing functions.** Consider the following definition:[2]

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is dominated by** f (or f **dominates** g) if and only if for every natural number n, $g(n) \leq f(n)$.

(a) Express this definition symbolically by showing how to define the following predicate:

$$Dom(f, g) : \underline{\hspace{6cm}}, \text{ where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}.$$

(b) Let $f(n) = 3n$ and $g(n) = n$. Prove that g is dominated by f.

(c) Let $f(n) = n^2$ and $g(n) = n + 165$. Prove that g is *not* dominated by f. Make sure to write the statement you'll prove in predicate logic, in fully simplified form (negations moved all the way inside).

[1] If you have time, you might want to consider whether the given statement is true or false, and write a correct proof or disproof.

[2] We'll use the symbol $\mathbb{R}^{\geq 0}$ to denote the set of all nonnegative real numbers, i.e., $\mathbb{R}^{\geq 0} = \{x \mid x \in \mathbb{R} \land x \geq 0\}$.

(d) Now let's *generalize* the previous statement. Translate the following statment into symbolic logic (expanding the definition of Dom) and then prove it!

For every positive real number x, $g(n) = n + x$ is *not* dominated by $f(n) = n^2$.

4. **More with floor**. Recall that the **floor** of a number x, denoted $\lfloor x \rfloor$, is the maximum integer less than or equal to x. We can always write $x = \lfloor x \rfloor + \epsilon$, where $0 \leq \epsilon < 1$.

   Prove the following statement:[3]

$$\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow \left(\lfloor x \rfloor\right)^2 \geq \frac{1}{2}x^2$$

   Hint: First, prove the following simpler statement, and use it in your proof: $\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$.

[3] For extra practice, try proving the following generalization of this statement: $\forall k \in \mathbb{R}^{\geq 0},\ k < 1 \Rightarrow \left(\exists x_0 \in \mathbb{R}^{\geq 0},\ \forall x \in \mathbb{R}^{\geq 0},\ x \geq x_0 \Rightarrow \left(\lfloor x \rfloor\right)^2 \geq kx^2\right)$.